DECOMPOSITIONS OF A T_1 SPACE

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Introduction. Several authors have proved theorems of the type: The "structure" of a certain class of transformations defined on a "suitable" space A to a fixed "suitable" space B "determines" the space A.

As examples, we have:

Banach [**3**, p. 170, see also **6**, **7**, **13**]:[1] The Banach space of all real, continuous functions defined on a compact metric space A "determines" A.

Eidelheit [**5**, see also **2**, **10**, **11**]: The ring of all bounded operators on a real Banach space A "determines" A.

In the present paper, we prove an analogous theorem (Theorem 2.5). Intuitively, it says that a T_1 space A is "determined" by a rather weak ordered system structure of the collection of all continuous mappings of A onto an arbitrary (variable) T_1 space B. More exactly, it states: If two T_1 spaces A, B are such that the ordered system of upper semi-continuous decompositions of A is isomorphic to that of B, then A and B are homeomorphic.

In §1 we give a discussion of ordered systems which is sufficient for our purposes. In §2 we prove the theorem mentioned above. In §3 we characterize separation and connectedness properties of a T_1 space in terms of order properties of its upper semi-continuous decompositions. In §4 we discuss compactness properties of the space and their relations to order properties of the decompositions, and in §5 we give some examples and counter examples.

1. Ordered systems.[2]

We assume that the reader is fairly familiar with the nomenclature of ordered systems and lattices.

An *ordered system* is a collection, M, of elements, D, with an ordering defined in M. That is, there is given a binary relation, $>$, which is defined for some pairs of elements in M, and which is transitive, reflexive, and proper.

We use *less than* and *greater than* to refer to the ordering in M; *contains* will be used only in the point set sense.

Most of the ordered systems we consider will be directed and will contain atoms. Throughout the paper, we use the letters a, b, c for

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[1] Numbers in square brackets refer to the bibliography at the end of the paper.

[2] For a more complete discussion, see [**4**, **12**, **14**].

atoms of ordered systems.

A *multiplicative system* is an ordered system in which each pair of elements has an inf; we use σ-complete and complete in the usual sense.

THEOREM 1.1. *A complete multiplicative system M with a unit is a complete lattice.*

For any non-empty collection $\{D_\alpha\}$ of elements of M, define[3]

$$D' = \wedge \{D \mid D > D_\alpha, \text{ all } \alpha\}.$$

Since M has a unit, $\{D \mid D > D_\alpha, \text{ all } \alpha\}$ is not empty, and D' exists; also, from the definition of $\wedge$, $D' > D_\alpha$ for all α. Suppose $D'' > D_\alpha$ for all α. Then

$$D' = \wedge \{D \mid D > D_\alpha\} = D'' \wedge (\wedge \{D \mid D > D_\alpha, D \neq D''\} < D''$$

so that $D' = \vee D_\alpha$ and M is a complete lattice.

Two ordered systems M, M' are *isomorphic* if there is a one-to-one, order preserving correspondence between them; that is, if there exists a one-to-one mapping $f(M) = M'$ such that $D' < D''$ is equivalent to $f(D') < f(D'')$.

Some examples of ordered systems which we shall use in the sequel are:

1. The collection of all open neighborhoods of a fixed point of any topological space, ordered by point set inclusion. That is, $U < V$ means $U \subset V$. This system is directed both by $>$ and by $<$ ($<$ is the "interesting" direction), in fact, it is a lattice. It contains a unit, but usually not a zero, and even if it has a zero, it may contain no atoms.

2. The lattice of all closed subsets of a T_1 space, again ordered by point set inclusion. Since this lattice is complete, it has a zero and a unit. The closed sets containing just one point are atoms, and these are the only atoms.

2. The reconstruction of S from M.

Throughout the paper, S will denote a topological space which contains an infinite number of points (four points would be sufficient), and in which each subset containing exactly one point is closed; that is, S is a T_1 space. By a decomposition of S we shall mean a collection of non-empty, closed, disjoint subsets of S whose union is S.

DEFINITION 2.1. *A decomposition of S is upper semi-continuous*

[3] We use $\vee$, $\wedge$ for sup, inf; $\cup$, $\cap$ for union, intersection, respectively; and $\{p \mid q\}$ for the set of all elements p with property q.

(u.s.c.) if, for any set X of the decomposition and any open set $U \supset X$, there exists an open set $V \supset X$ such that any set of the decomposition which meets V is contained in U.

It is well known that every upper semi-continuous decomposition (u.s.c.d.; the same abreviation will be used for the plural) of S gives rise to a continuous mapping of S onto a T_1 space—the Zerlegungsraum of Alexandroff and Hopf [1, p. 61 ff.]. Thus any statement concerning u.s.c.d. may be interpreted as a statement about continuous mappings of S onto a T_1 space.

Let $M = \{D\}$ be the ordered system of u.s.c.d. of S, where $D' < D''$ means each set of the decomposition D' is contained in some set of the decomposition D''. It is easy to see that M has a zero, 0, (the decomposition of S into the single points of S) and a unit, I, (the decomposition of S into the single set S) and is therefore directed both by $>$ and by $<$. However (Example 5.3), M is in general not a multiplicative system.

If x_a, y_a are any two distinct points of S, the decomposition of S which consists of the set $x_a \cup y_a$ and the single points of $S-(x_a \cup y_a)$ is u.s.c. and is an atom of M. Conversely, any atom of M has this form. We shall say that a is *generated* by x_a, y_a; the generators of an atom are unique, and completely determine the atom. Even though M is not necessarily a lattice, it is clear that $\bigvee a_i$ $(i=1, 2, \cdots, n)$ exists for any finite number of atoms and $a_i \wedge a_j = 0$ if $a_i \neq a_j$.

DEFINITION 2.2. *Two atoms a, b are concurrent, in symbols, $a \sim b$, if and only if it is false that $a \vee b$ is greater than exactly two atoms.*

THEOREM 2.1. *A necessary and sufficient condition that $a \sim b$ is that $(x_a \cup y_a) \cap (x_b \cup y_b) \neq 0$ (a; b generated by x_a, y_a; x_b, y_b respectively).*

NOTE. From this theorem, it will be observed that the relation $\sim$ is reflexive and symmetric but not, in general, transitive.

Sufficiency. We may assume $x_a = x_b$. Then either $y_a = y_b$ so that $a = b = a \vee b$ and $a \vee b$ is greater than just one atom, or $y_a \neq y_b$. In the latter case, $a \vee b$ is the decomposition of S into $x_a \cup y_a \cup y_b$ and the single points of $S-(x_a \cup y_a \cup y_b)$. But then there are three distinct atoms, namely a, b, and the atom generated by y_a, y_b, which are less than $a \vee b$.

Necessity. Suppose a and b are not concurrent, then $a \neq b$ since otherwise $a \vee b = a$ and is greater than just one atom. But we have seen above that if $a \neq b$ and $(x_a \cup y_a) \cap (x_b \cup y_b) \neq 0$, there are three atoms less than $a \vee b$. Therefore, since $a \vee b$ is greater than only two atoms by hypothesis, we must have

$$(x_a \cup y_a) \cap (x_b \cup y_b) = 0.$$

DEFINITION 2.3. *An u.s.c.d., D, of S is trivial if, for any two atoms a, b less than D, there exists an atom c less than D satisfying $a\sim c$, $b\sim c$.*

THEOREM 2.2. *An u.s.c.d., D, of S is trivial if and only if it consists of single points of S except for one closed set X (which may be empty, but cannot contain just one point). We say that D is generated by X, and use the notation D_X for it.*

Sufficiency. We may suppose $X\neq 0$, since otherwise $D=0$ and is trivial. If a, b are any two atoms less than D, the atom c generated by x_a, y_b (which will be distinct points under suitable labeling) satisfies our requirements.

Necessity. If there are two sets X, Y of the decomposition D, each containing at least two points, say $x_a, y_a \in X$, $x_b, y_b \in Y$, then a, b are two atoms less than D such that no atom c less than D satisfies $a\sim c$, $b\sim c$.

DEFINITION 2.4. *An infinite collection of atoms of M is a C-set if each pair of atoms of the collection are concurrent.*

Let us order the C-sets of M by inclusion, that is $C' > C''$ if each atom in C'' is also in C'. Then by Zorn's Lemma (or see below) there exist maximal C-sets. We denote the collection of all maximal C-sets of M by S'.

THEOREM 2.3. *There is a one-to-one correspondence between the points of S and the maximal C-sets of S'.*

Let x be any point of S and define $f(x)$ as the collection of all atoms generated by x, y for arbitrary $y\in S-x$. Then, by Theorem 2.1, $f(x)$ is a C-set. (It is infinite since S has an infinite number of points.) Moreover, it is a maximal C-set, for if C' is any C-set containing $f(x)$, and $c'\in C'$, then (Theorem 2.1) one of the generators of c' must be x, and $c'\in f(x)$. Thus $f(x)$ is a single-valued transformation of S into S'.

The inverse function f^{-1} is single-valued. For, if $f(x)=f(y)$, then any atom of M which has x as a generator also has y as a generator, and $x=y$.

$f(S)$ covers S'. For, if C is any maximal C-set, and a, b are any two distinct atoms in C, then $a\sim b$ and we may suppose $x_a=x_b$, $y_a\neq y_b$. Now, if c is any atom in C, we must have

$$(x_a \cup y_a) \cap (x_c \cup y_c) \neq 0 \neq (x_b \cup y_b) \cap (x_c \cup y_c)$$

so that (with suitable labeling) either (1) $x_c = x_a = x_b$, or (2) $x_c = y_a$ and $y_c = y_b$. But the latter case is impossible because C must contain an infinite number of atoms whereas if (2) holds no atom of $M-(a \cup b \cup c)$ can be concurrent with each of a, b, c. Thus (1) is the case, and $f(x_a) = C$.

THEOREM 2.4. *If M is the ordered system of all u.s.c.d. of a T_1 space S, then, using only the order properties of M, a T_1 space S' may be defined which is homeomorphic to S.*

The points of S' will be the maximal C-sets of M. A subset, Q, of S' will be closed if either (1) Q consists of exactly one point of S', or (2) there exists a trivial decomposition, D, of S such that Q is the collection of all maximal C-sets which contain at least one atom less than D.

It is easy to see that S' is a T_1 space. We prove that the transformation $f(S) = S'$ defined in the proof of Theorem 2.3 is a homeomorphism. We must only show that f and f^{-1} are closed.

Let X be any closed subset of S which contains more than one point. Then the closed subset of S' given by condition (2) above with $D = D_X$ is exactly $f(X)$.

Conversely, if X' is closed in S' and contains more than one point, and if D_X is the trivial decomposition given by condition (2) above, then $f^{-1}(X') = X$.

Since both S and S' are T_1 spaces, f and f^{-1} are closed and f is a homeomorphism.

THEOREM 2.5. *A necessary and sufficient condition that two T_1 spaces be homeomorphic is that their ordered systems of u.s.c.d. be isomorphic.*

The proof is immediate from Theorem 2.4.

3. Separation properties of S.

THEOREM 3.1. *If S is normal, M is a multiplicative system.*

Let $D' = \{X\}$, $D'' = \{Y\}$ be any two elements of M, and define

$$D = \{X \cap Y \mid X \in D', Y \in D'', X \cap Y \neq 0\}.$$

Clearly D is a decomposition of S; we show that it is u.s.c. If $X \cap Y \in D$ and U is any open set containing $X \cap Y$, then $X - U$, $Y - U$ are closed, disjoint subsets of S so that there exist open, disjoint subsets W', W'' containing them, respectively. Now $U \cup W' \supset X$, $U \cup W'' \supset Y$ and, since D', D'' are u.s.c., there exist open sets $V' \supset X$, $V'' \supset Y$ such that any set of $D^{(i)}$ which meets $V^{(i)}$ is contained in $U \cup W^{(i)}$ $(i = ', '')$.

Set $V = V' \cap V''$. Then, if $X' \cap Y'$ is any set of D which meets V, we have

$$X' \cap V' \neq 0, \quad \text{and} \quad X' \subset U \cup W',$$
$$Y' \cap V'' \neq 0, \quad \text{and} \quad Y' \subset U \cup W'',$$

so that

$$X' \cap Y' \subset (U \cup W') \cap (U \cup W'') = U,$$

and D is u.s.c. Clearly $D = D' \wedge D''$. Hence, M is a multiplicative system.

The converse of Theorem 3.1 does not hold. See Example 5.2.

DEFINITION 3.1. *Two trivial u.s.c.d. of S, D', D'' are concurrent, in symbols $D' \sim D''$, if some atom less than D' is concurrent with some atom less than D''.*

Notice that the trivial decomposition zero is not concurrent with any decomposition, and if D', D'' are atoms, Definition 3.1 agrees with Definition 2.2.

THEOREM 3.2. *A necessary and sufficient condition that two nonzero trivial decompositions D_X, D_Y be concurrent is that $X \cap Y \neq 0$.*

Sufficiency. Let $x \in X \cap Y$, $y \in Y - x$, $z \in X - x$. The atoms generated by x, z; x, y are less than D_X, D_Y respectively, and are concurrent, so that $D_X \sim D_Y$.

Necessity. If $X \cap Y = 0$, then no atom less than D_X can be concurrent with any atom less than D_Y, so that D_X and D_Y are not concurrent.

DEFINITION 3.2. *Two nonzero, trivial decompositions D', D'' run over M if each atom of M is concurrent with at least one of D', D''.*

THEOREM 3.3. *A necessary and sufficient condition that two nonzero, trivial decompositions, D_X, D_Y run over M is that $S - (X \cup Y)$ contains at most one point.*

If $S - (X \cup Y)$ contains two distinct points x, y, then the atom generated by x, y is not concurrent with either D_X or D_Y.

If $S - (X \cup Y)$ contains zero or one points, then any atom of M must have one of its generators in $X \cup Y$ and, by Theorem 3.2, must be concurrent with either D_X or D_Y.

THEOREM 3.4. *S is a Hausdorff space if and only if, for any atom $a \in M$, there exist trivial decompositions D', D'' which run over M, such that*

$$D' \sim a, \quad a \sim D'', \quad D' \not> a, \quad a \not< D''.$$

Suppose S is a Hausdorff space, and let a be any atom of M. Then x_a, y_a have disjoint neighborhoods U, V, and $X=S-U$, $Y=S-V$ form a closed covering of S. The trivial decompositions D_X, D_Y satisfy the conditions of the theorem.

Conversely, if x_a, y_a are any two distinct points of S, let a be generated by x_a, y_a, and let D_X, D_Y be the trivial decompositions satisfying the conditions of the theorem. Then (under suitable labeling)

$$x_a \in X \cap (S - Y), \qquad y_a \in Y \cap (S - X)$$

and $U=S-X$, $V=S-Y$ are neighborhoods of x_a, y_a respectively. If $S-(X\cup Y)$ is empty, $U\cap V=0$, if $S-(X\cup Y)$ contains just one point, z, then z is an isolated point and $U-z$, $V-z$ are disjoint neighborhoods of x_a, y_a, so that S is a Hausdorff space.

The proofs of the following theorems are similar to the proof of Theorem 3.4, and will not be given here.

THEOREM 3.5. *S is regular if and only if, for any nonzero trivial decomposition D, and any atom a not concurrent with D, there exist trivial decompositions D', D'' which run over M and such that $D'>a$, a is not concurrent with D'', D' is not concurrent with D, $D<D''$.*

THEOREM 3.6. *S is normal if and only if, for any two nonzero trivial decompositions D', D'', such that D' and D'' are not concurrent, there exist trivial decompositions D''', D'''' which run over M such that $D'''>D'$, D' is not concurrent with D'''', D''' is not concurrent with D'', $D''<D''''$.*

THEOREM 3.7. *S has at least one isolated point if and only if one of the following two equivalent conditions is fulfilled.*

(I) *There exists a trivial decomposition D, and an atom $a \not< D$ which run over M.*

(II) *There exists a nonzero trivial decomposition D and an atom $a \not< D$ such that, for any atom b, $b \not< D$ implies $b \sim a$.*

THEOREM 3.8. *S is connected, except possibly for one isolated point, if and only if every two trivial decompositions which run over M are concurrent.*

4. Compactness properties of S.

THEOREM 4.1. *If S is a compact[4] Hausdorff space, then M is a complete lattice.*

[4] A topological space is compact if every open covering has a finite subcovering or, equivalently, if every collection of closed subsets whose intersection is empty contains a finite subcollection with empty intersection.

By Theorem 1.1, we need only show that M is a complete multiplicative system. Let $\{D_\alpha\}$ be any non-empty collection of u.s.c.d. of S and set

$$D = \left\{ \bigcap_\alpha X_\alpha \mid X_\alpha \in D_\alpha, \bigcap_\alpha X_\alpha \neq 0 \right\}.$$

Clearly D is a decomposition of S; we show that it is u.s.c. If $X = \bigcap X_\alpha$ is any set of D and U is any open set containing X, then each of the sets $X_\alpha - U$ is closed and their intersection is empty. Since S is compact, there must be a finite number of the sets $X_\alpha - U$ whose intersection is empty, say

$$\bigcap_{i=1}^{n} (X_{\alpha_i} - U) = 0.$$

But S is normal, hence there exist open sets $W_i \supset X_{\alpha_i} - U$ $(i = 1, 2, \cdots, n)$ such that $\bigcap W_i = 0$. Now for each i, $U \cup W_i$ is an open set containing X_{α_i} and, since each D_α is u.s.c., there exist open sets $V_i \supset X_{\alpha_i}$ such that any set of D_{α_i} which meets V_i is contained in $U \cup W_i$. Set $V = \bigcap V_i$. Then $V \supset X$, and any set of D which meets V is contained in U. For, if $X' = \bigcap X'_\alpha$ meets V, then $X'_{\alpha_i} \cap V_i \neq 0$, and $X'_{\alpha_i} \subset U \cup W_i$, $i = 1, 2, \cdots, n$, so that

$$\bigcap_\alpha X'_\alpha \subset \bigcap_{i=1}^{n} X'_{\alpha_i} \subset \bigcap_{i=1}^{n} (U \cup W_i) = U.$$

Thus D is an u.s.c.d. of S. Clearly $D = \bigwedge D_\alpha$, and M is a complete lattice.

THEOREM 4.2. *If S is regular, and each set of the u.s.c.d. $D = \{X\}$ is compact, then $D \wedge D'$ exists in M for any $D' = \{Y\} \in M$.*

In the proof of Theorem 3.1, the only use which we made of the normality of S was in concluding that, since $X - U$, $Y - U$ were closed, disjoint subsets of S, they could be separated by disjoint open sets W', W''. Suppose X is compact and S is regular, then, for each point $x \in X - U$, there exist disjoint open sets V_x, W_x containing x, $Y - U$ respectively. Since X is compact, a finite number, say V_{x_i} $(i = 1, \cdots, n)$ of the V_x cover $X - U$ and

$$W' = \bigcup_i V_{x_i}, \qquad W'' = \bigcap_i W_{x_i}$$

are disjoint open sets containing $X - U$, $Y - U$ respectively.

THEOREM 4.3. *If S is a Hausdorff space and D', D'' are two u.s.c.d.*

of S such that each set of either decomposition is compact, then $D' \wedge D''$ exists in M.

The proof is similar to that for Theorem 4.2 and will not be given.

The author believes that the compactness hypotheses in Theorems 4.2, 4.3 are necessary, but has been unable to construct an example to prove their necessity. However, Example 5.3 shows that the separation hypotheses cannot be dropped.

THEOREM 4.4. *If S is metric, then a necessary and sufficient condition that M be a complete lattice is that S be either discrete or compact.*

By Theorem 4.1, if S is compact metric, M is a complete lattice; if S is discrete metric, then any decomposition of S is u.s.c. and M is a complete multiplicative system, hence, by Theorem 1.1, a complete lattice.

If S is metric but neither discrete nor compact, there exist two sequences of distinct points, x_i, y_i such that $\lim x_i = x$, $\{y_i\}$ has no limit point. Then the set

$$Y = x \cup (\bigcup x_i) \cup (\bigcup y_i)$$

is closed. Let a_i be the atom of M generated by x_i, y_i, then $D = \bigvee a_i$ does not exist in M.

For suppose $D \in M$. Consider the u.s.c.d. D_n $(n = 1, 2, \cdots)$ consisting of the sets (1) single points of $S - Y$, (2) the n sets $x_1 \cup y_1$, $x_2 \cup y_2$, $\cdots$, $x_n \cup y_n$, (3) the set $x \cup (\bigcup_{i>n} x_i) \cup (\bigcup_{i>n} y_i)$. Each $D_n > a_i$ for all i, hence $D < D_n$ for all n. Thus the decomposition D must have the single point x as one of its sets. But then D is not u.s.c. since each neighborhood of x meets all but a finite number of the sets $x_i \cup y_i$ of D, whereas the neighborhood $S - (\bigcup y_i)$ of x does not contain any of the sets $x_i \cup y_i$.

In this connection, see also Examples 5.1, 5.2, 5.4.

5. **Examples.** In this section we give several examples which have been referred to in the other sections of the paper. No proofs will be given.

EXAMPLE 5.1. S is the real line. M is not a σ-complete multiplicative system.

For each $n > 1$, let

$$D_n = \left\{ 1/2 \cup 2,\ 1/3 \cup 3,\ \cdots,\ 1/n \cup n,\ \left(\bigcup_{i>n} 1/i\right) \cup 0 \cup \left(\bigcup_{i>n} i\right), \text{ and single points of } S \text{ elsewhere} \right\}.$$

Then $\wedge D_n$ does not exist in M. See Theorem 4.4.

EXAMPLE 5.2. S is not discrete, not compact, not a Hausdorff space. M is a complete lattice.

Let S be the set of real numbers with "closed set" defined as (1) the whole space or (2) any countable (empty, finite, or denumerable) subset. Any decomposition of S is u.s.c. See Theorems 3.1, 4.4.

EXAMPLE 5.3. S is compact, not a Hausdorff space. M is not a multiplicative system.

Let S be the point set of the xy-plane consisting of the two line segments

$$L_i\colon\ y = i,\quad |x| \leqq 1, \qquad i = 1, 2,$$

but the topology of S is *not* its relative topology. The closed subsets of S are any finite union of the sets (1), single points of S, or (2), any subset of S which is closed in the topology of the xy-plane and contains its projection on L_i $(i=1, 2)$. (Intuitively, (2) says: any set which is the union of the "same" two "closed" subsets of the segments L_1, L_2.) Let

$$a_i = (-1/i, 0), \qquad b_i = (1 - 1/i, 0), \qquad i = 2, 3, \cdots,$$
$$Y = \bigcup (a_i \cup b_i) \cup (0, 0) \cup (0, 1) \cup (1, 0) \cup (1, 1),$$

and set

$$D' = \{\text{single points of } S - Y,\ (0, 0) \cup (1, 0),\ (0, 1) \cup (1, 1),\ (a_i \cup b_i)\ (i = 2, 3, \cdots)\},$$
$$D'' = \{\text{single points of } S - Y,\ (0, 0) \cup (1, 1),\ (0, 1) \cup (1, 0),\ (a_i \cup b_i)\ (i = 2, 3, \cdots)\}.$$

Then $D' \wedge D''$ does not exist in M. See Theorems 3.1, 4.1, 4.2, 4.3.

EXAMPLE 5.4. S is the xy-plane. M is not a lattice.

Let

$$Y = \bigcup_{m,n>0} (1/n, m/n),$$

and set

$$D' = \{\text{single points of } S - Y,\ (1/n, m/n) \cup (1/n, (m + 1)/n)\ (n = 1, 2, \cdots;\ m = 1, 3, 5, \cdots)\},$$
$$D'' = \left\{\text{single points of } (S - Y) \cup \left(\bigcup_n (1/n, 1/n)\right),\ (1/n, m/n) \cup (1/n, (m + 1)/n)\ (n = 1, 2, \cdots;\ m = 2, 4, 6, \cdots)\right\}.$$

Then $D'\vee D''$ does not exist in M. See Theorems 3.1, 4.1, 4.4.

In a future paper, the author will discuss an interesting related problem which was suggested to him by A. D. Wallace. Namely: Characterize intrinsically those ordered systems which are "M's" for some space S from a specified class, for example S compact Hausdorff.

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