# NEW SYSTEMS OF HYPERGEODESICS DEFINED ON A SURFACE

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**Introduction.** Let a non-ruled surface $S$ be referred to its asymptotic net as parametric. As a point $P_y$ moves along a curve $C_\lambda$ of $S$, the tangents at $P_y$ to the $u$- and $v$-asymptotic curves of $S$ describe two ruled surfaces $R_\lambda^u$ and $R_\lambda^v$, respectively. Let $S_\rho$ and $S_\sigma$ denote arbitrary transversal surfaces of the congruences of $u$- and $v$-tangents of $S$, respectively. The purpose of the present paper is to introduce and study systems of curves of $S$ which will be called $\rho$- and *$\sigma$-tangeodesics.*

DEFINITION. *A curve $C_\lambda$ of $S$ whose associated ruled surface $R_\lambda^u$ intersects the surface $S_\rho$ in an asymptotic curve of $R_\lambda^u$ is a $\rho$-tangeodesic of $S$. Similarly, a curve $C_\lambda$ of $S$ whose associated ruled surface $R_\lambda^v$ intersects $S_\sigma$ in an asymptotic curve of $R_\lambda^v$ is a $\sigma$-tangeodesic of $S$.*

The $\rho$- and *$\sigma$-tangeodesics* of $S$ at $P_y$ are found to be associated in remarkable manners with the *edges of Green*, the *directrices of Wilczynski*, and the *projective normal of Fubini*. In fact, a new geometric characterization is obtained for each of these lines.

1. **Tangeodesics.** If the parametric net on a non-ruled surface $S$ is the asymptotic net, the homogeneous projective coordinates $y^{(i)}(u, v)$ $(i=1, 2, 3, 4)$ of a general point $P_y$ of $S$ are solutions of a system of differential equations which may be assumed to be reduced to Wilczynski's canonical form

$$y_{uu} + 2by_v + fy = 0, \qquad y_{vv} + 2a'y_u + gy = 0. \tag{1.1}$$

The homogeneous coordinates of points $\rho$, $\sigma$ on arbitrarily selected transversal surfaces $S_\rho$ and $S_\sigma$ of the congruences of $u$- and $v$-tangents of $S$ are given by the vector forms

$$\rho = y_u - \beta y, \qquad \sigma = y_v - \alpha y, \tag{1.2}$$

wherein $\beta$, $\alpha$ are arbitrary analytic functions of $u$, $v$.

Let $l$ denote the line joining $\rho$, $\sigma$ and let $l'$ denote its reciprocal at $P_y$. The line $l'$ joins the points $P_y$ and $z$ where $z$ is given by

$$z = y_{uv} - \alpha y_u - \beta y_v \tag{1.3}$$

in which $\beta$ and $\alpha$ are the functions in (1.2). The line $l$, according to Green's classification, is a line of the first kind and generates a con-

Received by the editors February 20, 1943.

gruence $\Gamma$ of the first kind as $P_y$ moves over $S$. The line $l'$ is a line of the second kind and generates a congruence $\Gamma'$ of the second kind as $P_y$ moves over $S$.

Let $C_\lambda$ denote an integral curve of the curvilinear differential equation

$$(1.4) \qquad dv - \lambda(u, v)du = 0.$$

Regarding $u$ as independent variable we write $v' = \lambda(u, v)$ and $v'' = \lambda_u + \lambda\lambda_v$, in which accents indicate differentiation with respect to $u$.

The homogeneous coordinates of a general point of the ruled surface $R_\lambda^u$ are represented by the vector form

$$(1.5) \qquad \bar{y} = y_u + wy,$$

wherein $u$ and $w$ are independent variables and $v$ varies in accordance with the relation $v' = \lambda(u, v)$.

Let us put $u = u(t)$, $w = w(t)$, so that $\bar{y}$ describes a curve on $R_\lambda^u$ as $t$ varies. The necessary and sufficient condition that this curve be an asymptotic curve of $R_\lambda^u$ is that the determinant equation

$$(1.6) \qquad (\bar{y}, \bar{y}_u + v'\bar{y}_v, \bar{y}_w, d^2\bar{y}/dt^2) = 0$$

be satisfied. If we transform equation (1.6) by making use of equations (1.5), (1.4) and (1.1) we obtain, in view of the inequality $(y, y_u, y_v, y_{uv}) \neq 0$, the equation

$$(1.7) \qquad dw/du = \{2b^2 + (b_u - 2bw)v' + (w^2 + 2b_v + f)v'^2 - 2a'bv'^3 - bv''\}/v'^2.$$

As $P_y$ moves along $C_\lambda$ the point $\rho$ moves in the direction defined by (1.7) if and only if $w = -\beta$ satisfies (1.7). To obtain, therefore, the curvilinear differential equation for the $\rho$-tangeodesics we have merely to substitute $-\beta$ for $w$ in (1.7) and clear of fractions. The result is

$$(1.8) \qquad bv'' - 2b^2 - (2b\beta + b_u)v' - (\beta^2 + 2b_v + f + \beta_u)v'^2 + (2a'b - \beta_v)v'^3 = 0.$$

The differential equation for the $\sigma$-tangeodesics may be obtained by making the substitution

$$\begin{pmatrix} v'' & v' & v & b & a' & \beta & f \\ -v''/v'^3 & 1/v' & u & a' & b & \alpha & g \end{pmatrix}$$

in (1.8). The result, on simplifying, is

$$(1.9) \qquad a'v'' + \alpha_u - 2a'b + (\alpha^2 + 2a'_u + g + \alpha_v)v' + (2a'\alpha + a'_v)v'^2 + 2a'^2v'^3 = 0.$$

2. **Systems of hypergeodesics which have no cusp-axes.** The curves defined on a surface $S$ by a differential equation of the form

$$v'' = A + Bv' + Cv'^2 + Dv'^3 \tag{2.1}$$

in which the coefficients are functions of $u$, $v$ and accents indicate differentiation with respect to the independent variable $u$, are called *hypergeodesics.*[1] *The envelope of the osculating planes at a point $P_y$ of the hypergeodesics* (2.1) *is a cone which is ordinarily of the third class.* When this cone is of the third class it has three distinct cusp-planes which intersect in a line called *the cusp-axis of the cone*, or *the cusp-axis of the hypergeodesics at the point $P_y$*. The cusp-axis is the line $l'$ for which $\alpha$ and $\beta$ are given by

$$\alpha = C/2, \qquad \beta = -B/2. \tag{2.2}$$

We are interested here in those cases in which the class of the cone is less than three and the cone has no cusp-axis. The local equation of the osculating plane at $P_y$ of the curve $C_\lambda$ defined by (1.4) is

$$2\lambda(\lambda x_2 - x_3) + (\lambda' - 2b + 2a'\lambda^3)x_4 = 0, \tag{2.3}$$

when referred to the tetrahedron whose vertices have the general coordinates $y$, $y_u$, $y_v$, $y_{uv}$. Assuming $C_\lambda$ to be an integral curve of (2.1) we replace $\lambda'$ by the right member of (2.1) and put $\lambda$ in place of $v'$. The result is

$$2\lambda(\lambda x_2 - x_3) + (A - 2b + B\lambda + C\lambda^2 + [D + 2a']\lambda^3)x_4 = 0. \tag{2.4}$$

The *union curves of a congruence* $\Gamma'$ form a system of hypergeodesics, sometimes called an *axial system*, whose osculating planes at $P_y$ all pass through the line $l'$ of the congruence $\Gamma'$. Equation (2.1) represents such a system if $A = 2b$ and $D = -2a'$.

A system of hypergeodesics (2.1) for which

$$A = 2b, \qquad D \neq -2a' \tag{2.5}$$

will be called, for reasons which appear later, a *u-polar system*; and a system (2.1) for which

$$D = -2a', \qquad A \neq 2b \tag{2.6}$$

will be called a *v-polar system.*

If system (2.1) is a $u$-polar system, the equation for the envelope of its osculating planes at $P_y$ may be readily found from (2.4) to be

$$(2x_2 + Cx_4)^2 - 4(D + 2a')(Bx_4 - 2x_3)x_4 = 0. \tag{2.7}$$

Similarly, if (2.1) represents a $v$-polar system the equation for the envelope of its osculating planes at $P_y$ may be found to be

[1] G. Fubini, *Fondamenti della geometria proiettivo-differenziale di una superficie*, Atti Accad. Sci. Torino vol. 53 (1918) p. 1034.

$$(2.8) \qquad (2x_3 - Bx_4)^2 - 4(A - 2b)(2x_2 + Cx_4)x_4 = 0.$$

Since the cones (2.7) and (2.8) are nondegenerate quadric cones, they have no cusp-axes at $P_y$. Hence we have that *neither a u-polar system nor a v-polar system of hypergeodesics has a cusp-axis at* $P_y$.

There are two generators of the cone (2.7) which are such that the tangent planes of the cone along these generators pass through the $u$-tangent to $S$ at $P_y$. One of these is the $v$-tangent to $S$ at $P_y$ and the other is the line $l'$ for which

$$(2.9) \qquad \alpha = C/2, \qquad \beta = -B/2,$$

wherein $B$ and $C$ are the functions appearing in (2.7). This line $l'$ will be called the *u-edge* of the *u-polar system.*

The *v-edge* of a *v-polar system* is characterized similarly.

Since equations (2.9) are of the same form as equations (2.2), we have immediately this theorem.

THEOREM 2.1. *If the coefficients B and C of the differential equation of a non-polar system of hypergeodesics are identical with the corresponding coefficients of the differential equation of a u-polar system of hypergeodesics, the cusp-axis of the non-polar system at* $P_y$ *coincides with the u-edge of the u-polar system at* $P_y$.

A similar theorem applies, of course, to a $v$-polar system of hypergeodesics.

The forms of the differential equations (1.8) and (1.9) show clearly that the $\rho$- and $\sigma$-tangeodesics form $u$- and $v$-polar systems of hypergeodesics. For the system (1.8) we have

$$(2.10) \qquad \begin{gathered} A = 2b, \quad B = 2\beta + b_u/b, \quad C = (\beta^2 + 2b_v + f + \beta_u)/b, \\ D = (\beta_v - 2a'b)/b. \end{gathered}$$

For the system (1.9) we have

$$(2.11) \qquad \begin{gathered} A = (2a'b - \alpha_u)/a', \quad B = -(\alpha^2 + 2a_u' + g + \alpha_v)/a', \\ C = -2\alpha - a_v'/a', \quad D = -2a'. \end{gathered}$$

The cone (2.7) is associated with the system (1.8) of $\rho$-tangeodesics if $A$, $B$, $C$, $D$ are given by (2.10). Similarly, if $A$, $B$, $C$, $D$ are defined by (2.11), the cone (2.8) is associated with the system (1.9) of $\sigma$-tangeodesics.

The *u-edge* of the *ρ-tangeodesics* (1.8) is the line $l_1'$ passing through the points $P_y$ and $z_1$, where $z_1$ is given by $z_1 = y_{uv} - \alpha_1 y_u - \beta_1 y_v$, in which

$$(2.12) \qquad \beta_1 = -\beta - b_u/2b, \quad \alpha_1 = (\beta^2 + 2b_v + f + \beta_u)/2b.$$

The $v$-edge of the $\sigma$-tangeodesics is the line $l_2'$ passing through $P_y$ and $z_2$ where $z_2$ is given by $z_2 = y_{uv} - \alpha_2 y_u - \beta_2 y_v$, in which

$$(2.13) \quad \beta_2 = (\alpha^2 + 2a_u' + g + \alpha_v)/2a', \quad \alpha_2 = -\alpha - a_v'/2a'.$$

3. **The edges of Green, the directrices of Wilczynski, and the projective normal.** Let us apply the results of §§1 and 2 to obtain new characterizations of the *edges of Green*, the *directrices of Wilczynski*, and the *projective normal of Fubini*. The plane which is tangent to the cone (2.7) of the $\rho$-tangeodesics along the *u-edge* intersects the plane which is tangent to the cone (2.8) of the $\sigma$-tangeodesics along the *v-edge* in a line $\bar{l}'$ of the second kind which will be called the *joint-edge of the systems of $\rho$- and $\sigma$-tangeodesics of $S$ at $P_y$*. This line passes through the points $P_y$ and $\bar{z}$ where the general coordinates of $\bar{z}$ are given by $\bar{z} = y_{uv} - \bar{\alpha} y_u - \bar{\beta} y_v$, in which

$$(3.1) \qquad \bar{\alpha} = -\alpha - a_v'/2a', \qquad \bar{\beta} = -\beta - b_u/2b.$$

Since the functions $\alpha$, $\beta$ associated with the edges of Green are given by

$$(3.2) \qquad \alpha = -a_v'/4a', \qquad \beta = -b_u/4b,$$

we have the following theorem.

THEOREM 3.1. *The second edge of Green at $P_y$ lies in the plane $\pi$ determined by the joint-edge of the systems of $\rho$- and $\sigma$-tangeodesics of $S$ at $P_y$ and the reciprocal $l'$ of the line $l$ joining $\rho$, $\sigma$. The joint-edge coincides with the line $l'$ if and only if $l'$ is the second edge of Green. Any two particular planes $\pi_1$ and $\pi_2$ of the plane $\pi$ (corresponding to selections $\rho_1$, $\sigma_1$ and $\rho_2$, $\sigma_2$) intersect in the second edge of Green.*

Let $\sigma_a$ denote the intersection of the tangent plane to $S_\rho$ at $\rho$ with the $v$-tangent to $S$ at $P_y$ and let $\rho_a$ denote the intersection of the tangent plane to $S_\sigma$ at $\sigma$ with the $u$-tangent to $S$ at $P_y$. It may be easily verified that the general coordinates of $\rho_a$ and $\sigma_a$ are given by $\rho_a = y_u - \beta_a y$, $\sigma_a = y_v - \alpha_a y$, wherein $\beta_a$, $\alpha_a$ are given by

$$(3.3) \qquad \beta_a = -(g + \alpha_v + \alpha^2)/2a', \quad \alpha_a = -(f + \beta_u + \beta^2)/2b.$$

The line $l_a$ joining $\rho_a$, $\sigma_a$ was introduced by the author in a previous paper[2] and called *the asymptotic associate of the line $l$* joining $\rho$, $\sigma$.

The plane determined by the $v$-tangent of $S$ at $P_y$ and the $u$-edge of the $\rho$-tangeodesics at $P_y$ is the polar plane of the $u$-tangent of $S$ at $P_y$ with respect to the cone (2.7) of the $\rho$-tangeodesics. Similarly, the

[2] P. O. Bell, *A study of curved surfaces by means of certain associated ruled surfaces*, Trans. Amer. Math. Soc. vol. 46 (1939) p. 396.

plane determined by the $u$-tangent of $S$ at $P_y$ and the $v$-edge of the $\sigma$-tangeodesics at $P_y$ is the polar plane of the $v$-tangent of $S$ at $P_y$ with respect to the cone (2.8) of the $\sigma$-tangeodesics. These two polar planes intersect in a line $l_3'$ which will be called the *polar-axis of the $\rho$- and $\sigma$-tangeodesics at $P_y$*. This line may be shown to pass through the points $P_y$ and $z_3$ where the general coordinates of $z_3$ are given by $z_3 = y_{uv} - \alpha_3 y_u - \beta_3 y_v$, in which

$$\alpha_3 = -\alpha_a + b_v/b, \qquad \beta_3 = -\beta_a + a_u'/a'. \tag{3.4}$$

Since the functions $\alpha$, $\beta$ for the directrix $l'$ of Wilczynski are given by

$$\alpha = b_v/2b, \qquad \beta = a_u'/2a', \tag{3.5}$$

equations (3.4) are such that we have, immediately, this theorem.

THEOREM 3.2. *The second directrix of Wilczynski lies in the plane $p$ determined by the polar-axis of the $\rho$- and $\sigma$-tangeodesics at $P_y$ and the reciprocal $l_a'$, with respect to $S$ at $P_y$, of the asymptotic associate of $l$. Any two particular planes $p_1$ and $p_2$ of the plane $p$ (corresponding to lines $l_1$ and $l_2$) intersect in the second directrix of Wilczynski.*

Theorems 3.1 and 3.2 may be dualized by replacing the lines and planes of these theorems by their reciprocals with respect to $S$ at $P_y$. The dual of Theorem 3.1 is the following theorem.

THEOREM 3.3. *The first edge of Green contains the point $P$ of intersection of the reciprocal of the joint-edge of the systems of $\rho$- and $\sigma$-tangeodesics of $S$ and the line $l$ joining $\rho$, $\sigma$. These three lines coincide if and only if the line $l$ is the first edge of Green. Any two particular points $P_1$ and $P_2$ of the point $P$ (corresponding to selections $\rho_1$, $\sigma_1$ and $\rho_2$, $\sigma_2$) determine the first edge of Green.*

The statement of the dual of Theorem 3.2 will be left to the care of the reader.

Finally, since the *projective-normal* of $S$ at $P_y$ is the line for which the functions $\alpha$, $\beta$ are given by $\alpha = -(b_v/2b + a_v'/2a')$, $\beta = -(a_u'/2a' + b_u/2b)$, and the first directrix of $S$ at $P_y$ is the line $l$ for which $\alpha = b_v/2b$, $\beta = a_u'/2a'$, we have from equations (3.1) this theorem.

THEOREM 3.4. *If the line $l$ joining $\rho$, $\sigma$ is the first directrix of Wilczynski, the joint-edge of the systems of $\rho$ and $\sigma$-tangeodesics of $S$ at $P_y$ is the projective normal of Fubini.*

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