THE BETTI GROUPS OF SYMMETRIC AND CYCLIC PRODUCTS

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1. **Introduction.** Consider a finite complex K and a group of permutations of n elements $G=\{G_\lambda\}$, $\lambda=1, \cdots, N$. To define *the product* k^n *of* K *with respect to* G, $n=2, 3, \cdots$, we consider an ordered set of n complexes $K_1, \cdots, K_n$ each homeomorphic to K; here as throughout the paper we do not distinguish between a complex and a geometric realization of the complex. A point p of the topological product $K^n=K_1\times \cdots \times K_n$ can be represented by the sequence of points $p_1, \cdots, p_n$, $p_i\in K_i$. Each function $G_\lambda(p)$, $\lambda=1, \cdots, N$, gives a homeomorphism of K^n upon itself. We identify each point $p\in K^n$ with all its transforms $G_\lambda(p)$, $\lambda=1, \cdots, N$. The resulting continuous image of K^n is k^n. If G is the symmetric group or the cyclic group of permutations of n elements, the product k^n is called the *n-fold symmetric product* or the *n-fold cyclic product of* K, respectively.

In this paper we study the integral cohomology groups of k^n. Our Theorem 1 gives a convenient method for calculating these groups when G is given. The method is used to construct the cohomology groups when G is either symmetric or cyclic.

The method of this paper differs from that of the earlier papers [**3**] and [**5**] of the references at the end of this paper in the following way. All treatments consider Richardson's simplicial transformation Λ of K^n upon k^n. But Richardson and Walker use Λ to determine a transformation of cycles of K^n into cycles of k^n, while this paper considers the natural transformation of cocycles of k^n into cocycles of K^n. The earlier correspondence of cycles is not (1-1), but the present correspondence of cocycles is (1-1). This fact enables us to get new results.

2. **The general theorem.** By definition k^n is obtained by identifying points of K^n. This identification gives a continuous transformation Λ of K^n upon k^n. Richardson has shown[1] that K^n and k^n can be subdivided into simplicial complexes and the simplexes of these complexes so oriented that Λ is simplicial, G_λ is simplicial, $\lambda=1, \cdots, N$, and for any oriented simplex x of K^n

$$(1) \qquad \Lambda x = \Lambda G_\lambda x, \qquad \lambda = 1, \cdots, N.$$

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[1] See [**3**, §5].

Henceforth K^n and k^n shall denote these subdivisions.

We say that a chain F of K^n is *invariant under* G if $F(x)=F(G_\lambda x)$, $\lambda=1, \cdots, N$, for all simplexes x of K^n with the same dimension as F.

Let f be a chain of k^n, and let σf be the chain of K^n defined by $\sigma f(x)=f(\Lambda x)$.

THEOREM 1. *The transformation σ gives a* (1-1) *correspondence between the cocycles of k^n and the cocycles of K^n invariant under G, and a cocycle of k^n cobounds if and only if its corresponding invariant cocycle of K^n cobounds an invariant chain.*[2]

PROOF. To show that σf is invariant we have using (1) that $\sigma f(x)=f(\Lambda x)=f(\Lambda G_\lambda x)=\sigma f(G_\lambda x)$.

Next we show that any invariant chain F can be written σf. Indeed, because of (1) and the fact that F is invariant we can define a chain f of k^n by the equation $f(\Lambda x)=F(x)$. Then $\sigma f(x)=f(\Lambda x)=F(x)$.

Since ΛK^n covers k^n, it follows that σ is (1-1) between chains of k^n and invariant chains of K^n. To complete the proof of Theorem 1 it is sufficient to show that $\dot{f}=z$ implies $(\sigma f)^\cdot=\sigma z$, and conversely; the dot denotes the coboundary operator. It is well known that $\dot{f}=z$ implies $(\sigma f)^\cdot=\sigma z$.[3] Suppose $(\sigma f)^\cdot=\sigma z$. Then $z(\Lambda x)=\sigma z(x)=(\sigma f)^\cdot(x)=\sigma\dot{f}(x)=\dot{f}(\Lambda x)$.

3. **The topological product** K^n. In this section we state some properties of K^n which can be derived when $n>2$ in the same way that they have been derived when $n=2$.[4] Let

$$Z_i,\ z_j,\ f_j, \qquad i=1, \cdots, I,\ j=1, \cdots, J, \tag{2}$$

form a basis for the integral chains of K^n of all dimensions; furthermore, let (2) be such that the Z_i generate the cocycles that are independent of coboundaries, the Z_i and z_j generate the cocycles, and

$$\dot{f}_j = e_j z_j, \qquad j=1, \cdots, J, \tag{3}$$

are a complete set of coboundary relations for the cocycles of (2).[5] Corresponding to any set of non-negative integers $a_1, \cdots, a_I$, $b_1, \cdots, b_J$, $c_1, \cdots, c_J$ with $\sum a_i+\sum b_j+\sum c_j=n$ we have a chain $A=A(a_1, \cdots, a_I, b_1, \cdots, b_J, c_1, \cdots, c_J)$ defined as follows. Let

$$x_1, \cdots, x_n \tag{4}$$

[2] This theorem resembles [**4**, p. 22, line 15].

[3] See, for example, [**2**, chap. IV, §4].

[4] See, for example, [**2**] or [**1**].

[5] See [**1**, p. 304, §7], which includes a justification of the (1-1)-correspondence between the z's and f's of (2).

be the sequence of elements of (2) with Z_1 in the first a_1 places, Z_2 in the next a_2 places, z_1 in the b_1 places following the a_I elements equal to Z_I, f_1 in the c_1 places following the b_J elements equal to z_J, and so on. Then $A=(x_1\times \cdots \times x_n)$. If we denote the dimensions of Z_i and z_j by r_i and s_j, respectively, we see from (3) that the dimension of f_j is s_j-1. Hence the dimension of A is $\sum a_i r_i+\sum b_j s_j+\sum c_j(s_j-1)$.

Let $S=\{S_\lambda\}$, $\lambda=1, \cdots, n!$, be the symmetric group of permutations on n elements. We can apply S_λ to the sequence (4) and obtain the sequence which we denote by $y_1, \cdots, y_n$. We define $S_\lambda\{A\}=(y_1\times \cdots \times y_n)$. Then a basis for the chains of K^n is given by the distinct chains of the set $S_\lambda\{A\}$, $\lambda=1, \cdots, n!$, all A.

To obtain a basis for the cocycles of all dimensions we consider $\mathfrak{Z}_1=\mathfrak{Z}_1(a_1, \cdots, a_I, b_1, \cdots, b_J)=A(a_1, \cdots, a_I, b_1, \cdots, b_J)$, $\sum a_i+\sum b_j=n$. Also we consider $\mathfrak{Z}_2=\dot{A}/e$, $\sum c_j>1$, where e is the greatest common divisor of all the e_j's that are associated by (3) with the f_j's that correspond to the nonzero c_j's of A; the division of $\dot{A}$ by e can be shown to be always possible. Then a basis for the cocycles of K^n is given by the distinct chains of the set $S_\lambda\{\mathfrak{Z}_1\}$ and $S_\lambda\{\mathfrak{Z}_2\}$. $\lambda=1, \cdots, n!$, all $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$.

4. **The integral cohomology groups of the n-fold symmetric product.** We can consider the group S as the group G of §§1 and 2. Then any S_λ determines a simplicial map of K^n into itself. Under this simplicial map the chain A is mapped into a chain which we denote by $S_\lambda A$. From [3] we obtain the formula

$$S_\lambda A = (-1)^d S_\lambda\{A\} \tag{5}$$

where d is determined as follows. If S_λ interchanges two elements and leaves the other $n-2$ invariant, then d is the product of the dimensions of the two elements of (4) that are interchanged by S_λ. Since any S_λ is a product of permutations of the type just considered, the rule just stated determines d for any S_λ.

We next determine the chains of K^n that are invariant under S. First consider an A with at least one of its a_i, b_j, or c_j having the properties that it is greater than one and that the Z_i, z_j, or f_j with which it is associated is of odd dimension. Then (5) implies that there is an S_λ such that $S_\lambda A=-A$. This implies that any cocycle invariant under S is linearly independent of A and indeed of $S_\lambda A$, $\lambda=1, \cdots, n!$.

Next assume that no a_i, b_j, or c_j of A has the properties just considered. Then there are $\pi=a_1!a_2! \cdots b_1! \cdots c_1! \cdots$ values of λ for which $S_\lambda A=A$. From this fact and the fact that the $S_\lambda A$ are elements of a basis (because of (5) and the fact that the $S_\lambda\{A\}$ form a basis), we see that $\sum_\lambda S_\lambda A$, $\lambda=1, \cdots, n!$, is divisible by π but by no integer greater than π. Finally, we infer that a basis for the chains of K^n invariant under S is given by the distinct chains of the set $(1/\pi)\sum_\lambda S_\lambda A$, $\lambda=1, \cdots, n!$, where A ranges over all A any of whose factors Z_i, z_j, and f_j is of even dimension if the corresponding a_i, b_j, or c_j is greater than 1.

In the same way we deduce from the facts of §3 that a basis for the cocycles invariant under S is given as stated in Theorem 2 below.

We next find the coboundaries of chains invariant under S that are linearly dependent on $(1/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_1$. Suppose for $\mathfrak{Z}_1$ we have $b_1 \neq 0$. Then $\mathfrak{Z}_1$ is a product of n cocycles at least one of which is z_1. Replace the first z_1 in this product by f_1, $\dot{f}_1 = e_1 z_1$. Let D denote the resulting chain. Then $D' = (b_1/\pi)\sum_\lambda S_\lambda D$, $\lambda=1, \cdots, n!$, is invariant under S and is not a proper multiple of any other invariant chain. Since $(x_1 \times \cdots \times x_n)^\cdot = \sum_i \pm (x_1 \times \cdots \times \dot{x}_i \times \cdots \times x_n)$, $i=1, \cdots, n$,[4] and since $(S_\lambda F)^\cdot = S_\lambda \dot{F}$, we have $\dot{D}' = \pm (b_1 e_1/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_1$. This implies that $(b_1 e_1, \cdots, b_J e_J)(1/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_1$ cobounds a chain invariant under S; here as elsewhere we understand that the greatest common divisor of zero and a positive integer is that integer. Furthermore, examination of our basis for the chains invariant under S shows that multiples of this coboundary are the only multiples of $\mathfrak{Z}_1$ that can be linearly dependent upon a coboundary of a chain invariant under S.

The definition of $\mathfrak{Z}_2$ implies that $(e/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_2$, $\lambda=1, \cdots, n!$, cobounds a chain invariant under S. Furthermore, multiples of this coboundary are the only multiples of $(1/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_2$ that are dependent on coboundaries of chains invariant under S. We have proved this theorem.

THEOREM 2. *A basis for the cocycles of K^n invariant under S is given by the distinct chains of the set $(1/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_1$ and $(1/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_2$, $\lambda=1, \cdots, n!$, where $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ range over all $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ any of whose factors Z_i, z_j, and f_j has even dimension if the associated a_i, b_j, or c_j is greater than 1; furthermore, the cocycles invariant under S that cobound chains of K^n invariant under S are generated by $(b_1 e_1, \cdots, b_J e_J)$ $(1/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_1$ and $(e/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_2$.*

5. **The integral cohomology groups of the n-fold cyclic product.** Let $C = \{C_n^\mu\}$, $\mu=1, \cdots, n$, denote the group of the cyclic permutations of n elements, where C_μ^1 is the permutation that replaces each element except the first by its predecessor, and C_n^μ is the μth power of C_n^1. Let $B = q[x_1 \times \cdots \times x_p]$ denote the chain $(x_1 \times \cdots \times x_p \times x_1 \times \cdots \times x_p \times \cdots)$ of K^{pq}. Furthermore, whenever a chain of K^{pq} is written in this notation, it is understood that q is maximal.

As in §4 we can consider $C_n^\mu B$ and $C_n^\mu\{B\}$. These chains satisfy (5). In particular, $C_{pq}^p B = \delta C_{pq}^p\{B\}$, where $\delta = -1$ if q is even and $\sum_1^p r_i$ is odd, r_i = dimension of x_i, and $\delta = 1$ if either q is odd or $\sum_1^p r_i$ is even. This implies $\sum_\mu C_{pq}^\mu B = 0$, $\mu = 1, \cdots, pq$, if q is even and $\sum_1^p r_i$ is odd, and the same sum is divisible by q if q is odd or $\sum_1^p r_i$ is even.

A basis for the chains of K^n invariant under C is given by the distinct chains of the set $(1/q)\sum_\mu C_n^\mu B$, $\mu = 1, \cdots, n$, $pq = n$, q odd or $\sum_1^p r_i$ even, where the x_i range over the elements of the basis (2).

Let $Z_1 = (1/q)\sum_\mu C_n^\mu B$, q odd or $\sum_1^p r_i$ even, where the factors of B contain no f_j. If the factors of Z_1 contain no z_j, then Z_1 is linearly independent of coboundaries. Suppose the first factor x_1 of B is z_1, and $\dot{f}_1 = e_1 z_1$. Let E be the chain of K^n defined by $E = (f_1 \times x_2 \times \cdots \times x_p \times x_1 \times \cdots \times x_p \times x_1 \times \cdots \times x_p \times \cdots)$. We have that $E' = \sum_\mu C_n^\mu E$, $\mu = 1, \cdots, n$, is a chain invariant under C. Furthermore, E' is not divisible by any integer different from ± 1. We compute $\dot{E}' = e_1 \sum_\mu C_n^\mu B = e_1 q Z_1$. Let ϵ be the greatest common divisor of all the e_j's that are associated by (3) with the z_js that occur among the factors of B. We conclude that $\epsilon q Z_1$ is a coboundary.

Let $Z_2 = (1/eq)\sum_\mu C_n^\mu \dot{B}$, $\mu = 1, \cdots, n$, q odd or $\sum_1^p r_i$ even, where the factors of B contain at least two f_j's (possibly equal), and e is the greatest common divisor of the e_j's associated with the f_j's among these factors. In counting the factors of B we count each factor the number of times it is repeated due to the symmetry of B. We can prove this theorem.

THEOREM 3. *A basis for the cocycles of K^n invariant under C is given by the distinct chains Z_1 and Z_2; furthermore, the cocycles of K^n invariant under C that cobound chains invariant under C are generated by $\epsilon q Z_1$ and $e Z_2$.*

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