

# THE ELEMENTARY DIVISOR THEOREM FOR CERTAIN RINGS WITHOUT CHAIN CONDITION

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1. **Introduction.** The Elementary Divisor Theorem is known to hold in principal ideal rings and in rings with Euclidean algorithm.<sup>1</sup> It is an open question whether it holds in every *Prüfer ring*, that is, in every domain of integrity in which every ideal with finite basis is a principal ideal.<sup>2</sup> (A Prüfer ring can also be characterized as a domain of integrity in which the greatest common factor of any finite number of elements can be represented as a linear combination of these elements.) The Elementary Divisor Theorem will here be proved for what will be called "adequate" rings. They are special Prüfer rings, but not restricted by any equivalent to a chain condition, so that they comprise considerably more than just the principal ideal rings.<sup>3</sup>

2. **Definition of adequate rings.** Let  $R$  be a domain of integrity,  $a, b$  in  $R$ , and  $a \neq 0$ . By a *relatively prime part* of  $a$  with respect to  $b$ , written  $RP(a, b)$ , we shall understand a factor  $a_1$  of  $a$  such that, if  $a = a_1 \cdot a_2$ ,

(i)  $(a_1, b) = 1$ ,

(ii)  $(a_2, b) \neq 1$  for any non-unit factor  $a_3$  of  $a_2$ .<sup>4</sup>

$RP(a, b)$  may or may not exist; if it does, it is, in a sense, a largest factor of  $a$  that is relatively prime to  $b$ .

We now define  $R$  to be an *adequate ring* if

(i)  $R$  is a Prüfer ring,

(ii)  $RP(a, b)$  exists for all  $a, b$  in  $R$  with  $a \neq 0$ .

3. **Relationship to Prüfer rings and principal ideal rings.** By definition every adequate ring is a Prüfer ring. On the other hand, every

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<sup>1</sup> B. L. van der Waerden, *Moderne Algebra*, vol. 2, Berlin, 1931, p. 122. For further reference, see the papers by J. H. M. Wedderburn (J. Reine Angew. Math. vol. 167 (1932)), N. Jacobson (Ann. of Math. (2) vol. 38 (1937)), and O. Teichmüller (Preuss. Akad. Wiss. Sitzungsber. 1937).

<sup>2</sup> Rings of this kind were considered by H. Prüfer in *Untersuchungen über Teilbarkeitseigenschaften in Körpern*, J. Reine Angew. Math. vol. 168 (1932).

<sup>3</sup> My thanks are due to Professor Reinhold Baer for several helpful suggestions.

<sup>4</sup> Notation throughout this paper: " $(a, b, \dots)$ " for "greatest common factor of  $a, b, \dots$ ," " $a|b$ " for " $a$  is a factor of  $b$ ," " $a=b$ " for " $a$  equals  $b$  except possibly for a unit factor." (This last convention serves the purpose of replacing statements about principal ideals simply by statements about their generating elements.)

principal ideal ring is adequate; for in a ring of that kind we have unique prime power decomposition, and hence the existence of  $RP(a, b)$  for  $a \neq 0$  is obvious. An example of an adequate ring which is not a principal ideal ring is furnished by the set of integral functions with coefficients in a field  $F$ . That a ring of this type is a Prüfer ring has been proved in a previous paper.<sup>5</sup> The existence of  $RP(a, b)$  is an immediate consequence of the unique factorization theorem for integral functions (Theorem 6 of the same paper). Theorem 8 of that paper, finally, states that the ring is not a principal ideal ring.

4. **Six lemmas.** Now let  $R$  be an adequate ring. Among the properties of such a ring are those stated in the six lemmas below. The first four are well known, and hold in any Prüfer ring; sketches of their proofs have been included, however, for reasons of completeness.

LEMMA 1. *If  $f|ab$  and  $(f, a) = 1$ , then  $f|b$ .*

PROOF. Let  $hf + ka = 1$ ; then  $f|abk = b(1 - hf) = b - bhf$  and hence  $f|b$ .

LEMMA 2. *An element  $a$  has an inverse  $a^*$  mod  $b$ ,*

$$(1) \quad a \cdot a^* \equiv 1 \pmod{b},$$

*if and only if  $(a, b) = 1$ . Moreover,  $a^*$ , if it exists, is uniquely determined mod  $b$ .*

PROOF. If  $aa^* \equiv 1 \pmod{b}$ , then  $aa^* = 1 + bk$  and hence  $(a, b) = 1$ . Conversely, if  $(a, b) = 1$ , then  $h$  and  $k$  exist such that  $ha + kb = 1$ , and hence  $ah \equiv 1 \pmod{b}$  and  $a^* = h$ . Finally, if  $aa^* \equiv aa^{**} \equiv 1 \pmod{b}$ , then  $a^* \equiv a^*(aa^{**}) \equiv (aa^*)a^{**} \equiv a^{**} \pmod{b}$ .

LEMMA 3. *The congruences*

$$(2) \quad x \equiv a_i \pmod{m_i}, \quad i = 1, 2, \dots, n,$$

*have a simultaneous solution  $x$ , provided any two  $m_i$  are relatively prime. Moreover, the solution is uniquely determined mod  $m$ , where  $m = m_1 m_2 \cdots m_n$ .*

PROOF. Let  $m = m_i M_i$ . Since any two  $m_i$  are relatively prime it follows that  $(m_i, M_i) = 1$ . Hence, by Lemma 2, the  $M_i$  possess inverses  $M_i^*$  mod  $m_i$ . It is seen at once that

$$(3) \quad x \equiv a_1 M_1 M_1^* + \cdots + a_n M_n M_n^* \pmod{m}$$

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<sup>5</sup> Theorem 9 in *Divisibility properties of integral functions*, Duke Math. J. vol. 6 (1940) p. 351.

is a required solution, and the uniqueness is obvious.

LEMMA 4. Let  $a_1, a_2, \dots, a_n$  not be all zero, and

$$(4) \quad (a_1, a_2, \dots, a_n) = d;$$

then there exist elements  $a_{ij}$  ( $i=2, \dots, n; j=1, \dots, n$ ) such that

$$(5) \quad \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = d.$$

PROOF. If all but one  $a_k$  are zero, then  $d = a_k$ . Choose  $a_{2k} = \dots = a_{nk} = 0$ , and the unity matrix for the rest of the  $a_{ij}$ ; the resulting determinant will be  $d$  or  $-d$ , if  $-d$ , a change of one 1 into  $-1$  will put things right. If, on the other hand, no more than  $n-2$  of the  $a_k$  are zero, we can proceed by induction. For  $n=1$  the assertion is trivially correct. For  $n > 1$  let

$$(6) \quad d' = (a_2, \dots, a_n);$$

by induction we can find  $a_{ij}$  ( $i=2, \dots, n-1; j=2, \dots, n$ ) such that

$$(7) \quad \begin{vmatrix} a_2 & \cdots & a_n \\ a_{22} & \cdots & a_{2n} \\ \cdot & \cdots & \cdot \\ a_{n-1,2} & \cdots & a_{n-1,n} \end{vmatrix} = d'.$$

Suppose now that

$$(8) \quad d = (a_1, d') = h \cdot a_1 + k \cdot d'.$$

We can then complete our determinant as follows:

$$(9) \quad \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & a_{n-1,2} & \cdots & a_{n-1,n} \\ (-1)^{n-1}k & (-1)^n h a_2 / d' & \cdots & (-1)^n h a_n / d' \end{vmatrix} = h \cdot a_1 + k \cdot d' = d.$$

LEMMA 5.  $RP(a, b)$  is uniquely determined except for a unit factor.

PROOF. Let

$$(10) \quad a = a_1 \cdot a_2 = a' \cdot a'',$$

where  $a_1$  and  $a'$  are both largest factors of  $a$  that are relatively prime to  $b$ . We show first that  $(a_2, a') = 1$ . Suppose that  $(a_2, a') = d \neq 1$ ; then  $d | a_2$ , and by definition of  $a_1$ ,  $(d, b) \neq 1$ ; but  $d | a'$  implies  $(d, b) | (a', b)$ , so that  $(a', b) \neq 1$ , which is impossible by definition of  $a'$ . Hence  $(a_2, a') = 1$ , and we can now apply Lemma 1: since  $a_2 | a'a''$ , it follows that  $a_2 | a''$ . Similarly  $a'' | a_2$ , so that  $a_2$  and  $a''$  can only differ by a unit factor.

LEMMA 6. *Let  $RP(a, b) = r$ , and let  $d | a$  with  $(d, r) = 1$ ; then  $RP(d, b) = 1$ .*

PROOF. Let  $a = r \cdot s$ . By assumption and Lemma 1,  $d | s$ . Hence, by definition of  $r$  and  $s$ ,  $(\delta, b) \neq 1$  for every non-unit factor  $\delta$  of  $d$ , and consequently  $RP(d, b) = 1$ .

5. **A preparatory theorem.** The main result of this paper will be a consequence of the following theorem.

THEOREM 1. *Let*

$$(11) \quad M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{pmatrix}$$

*be a matrix with coefficients in an adequate ring  $R$ , let*

$$(12) \quad 1 < n = \text{rank } M \leq k$$

*and*

$$(13) \quad (a_{11}, a_{12}, \cdots, a_{1k}, a_{21}, \cdots, a_{nk}) = d;$$

*then there exist  $t_1, t_2, \cdots, t_{n-1}$  in  $R$  such that*

$$(14) \quad (A_1, A_2, \cdots, A_k) = d,$$

*where*

$$(15) \quad A_i = a_{1i}t_1 + a_{2i}t_2 + \cdots + a_{n-1,i}t_{n-1} + a_{ni}, \quad i = 1, 2, \cdots, k.$$

PROOF. (a) Suppose the theorem has been proved for  $d = 1$ . Then the general case results as follows. Let

$$(16) \quad N = (1/d) \cdot M;$$

the coefficients

$$(17) \quad b_{ij} = a_{ij}/d$$



$$(27) \quad t \equiv a_{1i}^* \cdot (1 - a_{2i}) \pmod{D_i}, \quad i = 1, 2, \dots, k,$$

where  $a_{1i}^*$  is an inverse of  $a_{1i} \pmod{D_i}$ :

$$(28) \quad a_{1i}^* \cdot a_{1i} \equiv 1 \pmod{D_i}$$

(the existence of  $a_{1i}^*$  being guaranteed by Lemma 2, since  $(a_{1i}, D_i) = 1$  by virtue of (25)). With  $t$  chosen thus, we shall now show that

$$(29) \quad (A_1, A_2, \dots, A_k) = 1.$$

From (21) and (27) we have

$$(30) \quad A_i = a_{1i}t + a_{2i} \equiv 1 \pmod{D_i}.$$

Suppose, now, that the  $A_i$  have some non-unit factor  $c$  in common; then, by (24), it would follow that  $c \mid D$ . This, together with (30), implies that

$$(31) \quad (c, D_i) = 1;$$

this means that  $c$  is a factor of  $D$  that is prime to all  $D_i$ , and we can apply Lemma 6, with the result that

$$(32) \quad RP(c, a_{1i}) = 1, \quad i = 1, 2, \dots, k.$$

But then, as  $c$  was not supposed to be a unit, none of its non-unit factors is prime to any  $a_{1i}$ . In particular,

$$(33) \quad \begin{cases} (c, a_{11}) = c_1 \neq 1, \\ (c_1, a_{12}) = c_2 \neq 1, \\ \dots \dots \dots \dots \dots \dots, \\ (c_{k-1}, a_{1k}) = c_k \neq 1; \end{cases}$$

the  $a_{1i}$  thus have a common non-unit factor  $c_k$ , and, since  $c_k \mid A_i$ , it follows that  $c_k$  is also a factor of  $A_i - a_{1i}t = a_{2i}$ . But this is incompatible with the assumption that  $d = 1$ .

(c) Suppose now that the theorem has been proved for values less than  $n$ ; we then have to show that it holds for  $n$  provided  $d = 1$ . Let

$$(34) \quad d_1 = (a_{21}, a_{22}, \dots, a_{2k}, a_{31}, \dots, a_{nk});$$

applying induction, we can find  $t_2, t_3, \dots, t_{n-1}$  so that

$$(35) \quad (B_1, B_2, \dots, B_k) = d_1$$

where

$$(36) \quad B_i = a_{2i}t_2 + a_{3i}t_3 + \dots + a_{n-1,i}t_{n-1} + a_{ni}, \quad i = 1, 2, \dots, k,$$

and it remains to determine  $t_1$  suitably. We have

$$(37) \quad A_i = a_{1i}t_1 + B_i$$

with

$$(38) \quad (a_{11}, B_1, a_{12}, B_2, \dots, a_{1k}, B_k) = (a_{11}, a_{12}, \dots, a_{1k}, d_1) \\ = (a_{11}, a_{12}, \dots, a_{1k}, a_{21}, \dots, a_{nk}) = 1.$$

Therefore, using induction with  $n=2$ , we can find  $t_1$  such that

$$(39) \quad (A_1, A_2, \dots, A_k) = 1,$$

and the proof of Theorem 1 is completed.

**6. The case of a square matrix with nonzero determinant.** Before taking up the general case of the Elementary Divisor Theorem we shall first treat the case of a nonsingular matrix.

**THEOREM 2.** *Let  $R$  be an adequate ring,  $M$  a square matrix with coefficients in  $R$ , and  $|M| \neq 0$ . Then there exist two matrices  $X$  and  $Y$  with coefficients in  $R$ , such that*

$$(40) \quad |X| = |Y| = 1$$

and

$$(41) \quad XMY = \begin{pmatrix} e_1 & 0 & \cdots & 0 \\ 0 & e_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & e_n \end{pmatrix},$$

where the  $e_i$  are the elementary divisors of  $M$ .

**PROOF.** For  $n=1$  the theorem is trivially true. For  $n>1$  assume the theorem proved for all values less than  $n$ . It is sufficient to consider the case where  $e_1$  (which is the greatest common factor of the coefficients of  $M$ ) equals 1; for otherwise we first deal with the matrix  $M/e_1$ , whose elementary divisors are  $1, e_2/e_1, \dots, e_n/e_1$ ; hence if

$$(42) \quad X \frac{M}{e_1} Y = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & e_2/e_1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & e_n/e_1 \end{pmatrix},$$

we have at once

$$(43) \quad XMY = \begin{pmatrix} e_1 & 0 & \cdots & 0 \\ 0 & e_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & e_n \end{pmatrix}.$$

Let now  $e_1=1$  and  $M=(a_{ij})$ . Applying Theorem 1, with  $k=n$  and  $d=1$ , we can find  $t_1, t_2, \dots, t_{n-1}$  in  $R$  such that

$$(44) \quad (A_1, A_2, \dots, A_n) = 1,$$

where

$$(45) \quad A_i = a_{1i}t_1 + a_{2i}t_2 + \cdots + a_{n-1,i}t_{n-1} + a_{ni}, \quad i = 1, 2, \dots, n.$$

Applying Lemma 4 to these  $A_i$ , a square matrix  $A$  with coefficients in  $R$  can be found whose first row is  $A_1, A_2, \dots, A_n$  and whose determinant is 1. If  $h_i$  is the minor belonging to  $A_i$ , we have

$$(46) \quad h_1A_1 + h_2A_2 + \cdots + h_nA_n = 1.$$

As  $|A| = 1$ , we can form, in  $R$ , the inverse matrix

$$(47) \quad Q = A^{-1}.$$

Note that the first column of  $Q$  is  $h_1, h_2, \dots, h_n$ . If, now, we set

$$(48) \quad q_i = h_1a_{i1} + h_2a_{i2} + \cdots + h_na_{in}, \quad i = 1, 2, \dots, n,$$

we can define

$$(49) \quad P = P_1 \cdot P_2,$$

where

$$(50) \quad P_1 = \begin{pmatrix} 1 & 0 \cdots 0 \\ q_1 & 0 \cdots -1 \\ \cdot & \cdot \cdots \cdot \\ q_{n-1} & -1 \cdots 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} t_1 \cdots t_{n-1} & 1 \\ 0 \cdots 1 & 0 \\ \cdot \cdots \cdot & \cdot \\ 1 \cdots 0 & 0 \end{pmatrix}.$$

Then

$$(51) \quad |P_1| = |P_2| = (-1)^{n(n-1)/2}$$

so that

$$(52) \quad |P| = |P_1| \cdot |P_2| = 1.$$

Multiplying out, we obtain

$$(53) \quad P = \begin{pmatrix} t_1 & t_2 & \cdots & t_{n-1} & 1 \\ q_1 t_1 - 1 & q_1 t_2 & \cdots & q_1 t_{n-1} & q_1 \\ q_2 t_1 & q_2 t_2 - 1 & \cdots & q_2 t_{n-1} & q_2 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ q_{n-1} t_1 & q_{n-1} t_2 & \cdots & q_{n-1} t_{n-1} - 1 & q_{n-1} \end{pmatrix}.$$

Notice that we also have

$$(54) \quad |Q| = 1.$$

We now multiply  $M$  from the left by  $P$  and from the right by  $Q$ :

$$(55) \quad PMQ = \begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ q_1 A_1 - a_{11} & q_1 A_2 - a_{12} & \cdots & q_1 A_n - a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ q_{n-1} A_1 - a_{n-1,1} & q_{n-1} A_2 - a_{n-1,2} & \cdots & q_{n-1} A_n - a_{n-1,n} \end{pmatrix} \\ \cdot \begin{pmatrix} h_1 & * & \cdots & * \\ h_2 & * & \cdots & * \\ \cdot & \cdot & \cdots & \cdot \\ h_n & * & \cdots & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \cdot & \cdot & \cdots & \cdot \\ 0 & * & \cdots & * \end{pmatrix}$$

where the zeros in the first row of this matrix arise from multiplying the row  $A_1, A_2, \dots, A_n$  by a row of minors of  $A$  not belonging to that row, while the zero in the  $(i+1)$ st place of the first column results as follows:

$$(56) \quad \begin{aligned} &(q_i A_1 - a_{i1})h_1 + \cdots + (q_i A_n - a_{in})h_n \\ &= q_i(A_1 h_1 + \cdots + A_n h_n) - (a_{i1}h_1 + \cdots + a_{in}h_n) \\ &= q_i \cdot 1 - q_i = 0. \end{aligned}$$

We can now apply induction to the matrix  $M_1$  of order  $n-1$  that appears in the lower right corner of  $PMQ$ . Let

$$(57) \quad X_1 M_1 Y_1 = \begin{pmatrix} e'_1 & 0 & \cdots & 0 \\ 0 & e'_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & e'_{n-1} \end{pmatrix}$$

where  $|X_1| = |Y_1| = 1$  and where  $e'_1, e'_2, \dots, e'_{n-1}$  are the elementary divisors of  $M_1$ . Then, if we set

$$(58) \quad X = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & X_1 & & \\ 0 & & & \end{pmatrix} \cdot P, \quad Y = Q \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & Y_1 & & \\ 0 & & & \end{pmatrix},$$

we obtain immediately

$$(59) \quad XMY = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & e'_1 & 0 & \cdots & 0 \\ 0 & 0 & e'_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & e'_{r-1} \end{pmatrix},$$

and all that remains to be shown is that

$$(60) \quad e_i = e'_{i-1}, \quad i = 2, 3, \cdots, n.$$

Now, clearly, the determinant divisors of  $XMY$  are  $1, e'_1, e'_1 e'_2, \cdots, e'_1 e'_2 \cdots e'_{n-1}$ ; hence its elementary divisors are  $1, e'_1, e'_2, \cdots, e'_{n-1}$ , and the rest follows from the fact that "equivalent" matrices (here  $M$  and  $XMY$ ) have the same elementary divisors (proof literally as for the ring of rational integers).

**7. The general case.** We can now prove the general Elementary Divisor Theorem:

**THEOREM 3.** *Let  $R$  be an adequate ring, and  $M$  a matrix of type  $(m, n)$  with coefficients in  $R$ . Then there exist two matrices  $X$  and  $Y$  with coefficients in  $R$  and of types  $(m, m)$  and  $(n, n)$  respectively, such that*

$$(61) \quad |X| = |Y| = 1$$

and

$$(62) \quad XMY = \begin{pmatrix} e_1 & 0 & \cdots & 0 \\ 0 & e_2 & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & & e_r & \cdot \\ \cdot & & & 0 & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & \cdot & \cdots & & & 0 \end{pmatrix}$$

where  $e_1, e_2, \cdots, e_r$  are the elementary divisors of  $M$ .

PROOF. (a) Let  $M = (a_{ij})$ , of rank  $r$ . If  $r = 0$ , we can set  $X = E_m$  and  $Y = E_n$ . Otherwise not all  $a_{ij}$  are zero, and we have

$$(63) \quad (a_{11}, a_{12}, \dots, a_{1n}, a_{21}, \dots, a_{mn}) = d \neq 0.$$

The case where  $m = n = r$  is covered by Theorem 2. If  $r < m$  or  $r < n$  or both, it will be shown below that  $M$  can be transformed unitarily into a matrix of the form

$$(64) \quad \begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix}$$

where  $N$  is of type  $(r, r)$  (and hence  $|N| \neq 0$ ). Once this has been done, Theorem 2 can be applied to  $N$ ; and if, say,  $X_1$  and  $Y_1$  take  $N$  into the desired normal form, then

$$(65) \quad X = \begin{pmatrix} X_1 & 0 \\ 0 & E_{m-r} \end{pmatrix}, \quad Y = \begin{pmatrix} X_2 & 0 \\ 0 & E_{n-r} \end{pmatrix}$$

will do the same for  $M$ .

(b) Let  $r < m$ . The  $m$  rows of  $M$  must then be linearly dependent in  $R$ :

$$(66) \quad j_1 a_{1i} + j_2 a_{2i} + \dots + j_m a_{mi} = 0, \quad i = 1, 2, \dots, n,$$

where the  $j_i$  may be assumed to be relatively prime. By Lemma 4 we can find elements  $j_{ik}$  in  $R$  ( $i = 1, \dots, m-1; k = 1, \dots, m$ ) such that the matrix

$$(67) \quad J = \begin{pmatrix} j_{11} & j_{12} & \dots & j_{1m} \\ \cdot & \cdot & \dots & \cdot \\ j_{m-1,1} & \cdot & \dots & j_{m-1,m} \\ j_1 & j_2 & \dots & j_m \end{pmatrix}$$

has determinant 1; and if we multiply  $M$  on the left by  $J$ , we obtain a matrix whose last row consists of zeros.

Clearly, if  $r < n$ , an analogous multiplication of  $M$  on the right by a unitary matrix will generate a last column of zeros.

This procedure can be applied repeatedly. If

$$(68) \quad X_1 M X_2 = \begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix}$$

where  $M_1$  is of type  $(m_1, n_1)$  with  $r < m_1$  or  $r < n_1$ , a further row or column of  $M_1$  (and hence of  $M$ ) can be turned into zeros. This process will break off only when the form

$$(69) \quad \begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix}$$

with  $N$  of type  $(r, r)$  has been reached. This, according to (a), completes the proof.

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## THE GEOMETRY OF VELOCITY SYSTEMS

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**1. Introduction.** In connection with the investigation of the differential geometric properties of positional fields of force, Kasner introduced certain important systems of curves which he termed *velocity systems*.<sup>1</sup> In this paper, we propose to present some of the old and many new geometric properties of any velocity system.

Since velocity systems serve to characterize the conformal group, many of our results are logically theorems of conformal geometry. This may be contrasted with the study of all the trajectories of positional fields of force, which is essentially differential projective geometry.

**2. Velocity systems.** We shall find it most convenient for our study to use the minimal coordinates  $u = x + iy$ ,  $v = x - iy$ ,  $p = dv/du$ ,  $q = d^2v/du^2$ , where, of course,  $(x, y)$  are the cartesian coordinates of the plane.

In the plane, consider a particle of unit mass moving in any field of force whose components parallel to the  $u$ -axis and the  $v$ -axis are  $d(u, v)$  and  $c(u, v)$ . The equations of motion are

$$(1) \quad \frac{d^2u}{dt^2} = d(u, v), \quad \frac{d^2v}{dt^2} = c(u, v),$$

where  $t$  is the time. Now if  $r$  is the radius of curvature along a trajectory, we have

$$(2) \quad \omega^2 = rN,$$

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<sup>1</sup> Kasner, *Differential-geometric aspects of dynamics*, Amer. Math. Soc. Colloquium Publications vol. 3 (1912); Trans. Amer. Math. Soc. vols. 9-10 (1908-1909).