EXPANSIONS OF QUADRATIC FORMS

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1. **The problem.** A quadratic form Q with coefficients in a field K, whose characteristic is different from 2, is usually given as a linear combination

$$\sum_{i=1}^{n} a_{ij}x_ix_j \tag{1}$$

of products $\{x_i x_j\}$, where (a_{ij}) is symmetric. The sum (1) is one of the type

$$\sum_{i=1}^{\tau} L_iM_i, \tag{2}$$

where the L's and M's are linear forms. In general the decomposition (1) is not the most economical way of writing Q as a sum of the type (2) in the sense that τ is a minimum for Q. In treating algebras associated with quadratic forms E. Witt[1] showed that the form Q is equivalent under a nonsingular linear transformation to a decomposition

$$\sum_{i=1}^{\sigma} y_iz_i + \sum_{i=1}^{r-2\sigma} \nu_iu_i^2, \tag{3}$$

where the last sum is a nonzero form, and r is the rank of Q. In the present paper we shall show that the minimum τ for Q is $r-\sigma$. Thus this minimum τ is determined by the rank r and the "characteristic" σ of Q. This characteristic[2] is the maximum number σ of linearly independent linear forms $L_1, \cdots, L_\sigma$ such that the rank of $Q+\lambda_1L_1^2+\cdots+\lambda_\sigma L_\sigma^2$ is the same as the rank of Q for all values of the λ's. The form Q has characteristic σ *if and only if* Q has the *canonical splitting* $G+H$, where G has characteristic σ and rank 2σ, while H has characteristic 0 and rank $r-2\sigma$. The form G has a decomposition (2) with $\tau=\sigma$. The decomposition (3) is one such that the first sum is a form G of the type described and the other a form H. Thus it will be proved

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[1] E. Witt, *Theorie der Quadratischen Formen in beliebigen Körpern*, J. Reine Angew. Math. vol. 176 (1937) p. 35.

[2] Rufus Oldenburger, *The index of a quadratic form for an arbitrary field*, Bull. Amer. Math. Soc. abstract 48-5-162.

that the decomposition (3) corresponding to a canonical splitting $G+H$ of Q is one with a minimum number of terms.

Like the rank of Q the characteristic of Q has the property that this characteristic changes at most by 1 under addition of a term λL^2, L linear, to Q. We shall prove here that actually the minimum τ, the characteristic σ defined above, and the index[3] (if K is real) possess this property of changing at most by 1 under additions of the type LM to Q, where L and M are arbitrary linear forms.

We recall that the rank r of Q is the minimum τ for which Q can be written as a sum (2), where for each i the forms L_i and M_i are linearly dependent. Thus both the rank alone, and the rank and characteristic of Q, yield minimum properties of expansions of Q invariant under nonsingular linear transformations on the variables in Q.

It will be understood throughout the present paper that the coefficients are in a field K of the type specified above. The field K is otherwise unrestricted, except where K is taken to be the real or complex fields.

2. **Solution of the minimum problem.** The following lemma needs no proof.

LEMMA 1. *The characteristic of a quadratic form Q is invariant under nonsingular linear transformations on Q.*

The lemma to follow was proved elsewhere.[4]

LEMMA 2. *The characteristic of a quadratic form Q changes at most by* 1 *under addition to Q of a term λL^2, L linear and λ in the given field.*

LEMMA 3. *The characteristic of a quadratic form Q is at least as great as the characteristic of each form Q^* obtained from Q by imposing homogeneous linear relations on the variables in Q.*

We write Q as in (1), and suppose that Q^* is obtained from Q by equating $x_1, \cdots, x_{e-1}$ to zero for some e. The matrix $A=(a_{ij})$ of Q can be written as

$$\left\| \begin{matrix} * & * \\ * & B \end{matrix} \right\|,$$

where B is the matrix (a_{ij}) $[i, j=e, \cdots, n]$ of Q^*, and the asterisks in A indicate minors of A. We let α designate the characteristic of

[3] The *index* of a real quadratic form Q is the number h of $+$ signs in a canonical form $x_1^2+\cdots+x_h^2-x_{h+1}^2-\cdots-x_r^2$ to which Q is equivalent.

[4] See the above reference to a paper by R. Oldenburger.

the form Q^*. Since Q^* has a canonical splitting, as described in §1, the form Q^* is equivalent under a nonsingular transformation to the form

$$\sum_{i=1}^{\alpha} u_i v_i + F, \tag{4}$$

where F is a form of characteristic 0 whose variables are linearly independent of the u's and v's. With the form (4) we may associate the symmetric matrix

$$\begin{Vmatrix} 0 & 0 \\ 0 & C \end{Vmatrix}$$

of order $n-e+1$, where C is a nonsingular minor of the type

$$\begin{Vmatrix} 0 & 0 & I \\ 0 & D & 0 \\ I & 0 & 0 \end{Vmatrix},$$

the minor I being an identity matrix of order α [arising from the summation in (4)]. It follows from elementary matrix considerations that there is a nonsingular matrix N such that

$$N'AN = \begin{Vmatrix} E & 0 \\ 0 & 0 \end{Vmatrix},$$

where E is a nonsingular minor with the shape

$$E = \begin{Vmatrix} * & * & * & I \\ * & * & * & 0 \\ * & * & D & 0 \\ I & 0 & 0 & 0 \end{Vmatrix},$$

and N' designates the transpose of N. We let $T=T(y_1, \cdots, y_n)$ be a quadratic form in $y_1, \cdots, y_n$ associated in the usual manner with $N'AN$. The rank of $T+\lambda_1 y_1^2+ \cdots +\lambda_\alpha y_\alpha^2$ is the same as the rank of T for all values of the λ's. Since T is equivalent to Q the characteristic of Q is at least α.

Suppose now that we impose homogeneous linear relations $Z_1=0, \cdots, Z_s=0$ on the variables $x_1, \cdots, x_n$ in Q. It is no restriction to take these Z's to be linearly independent forms. We may therefore use these Z's and enough of the x's to obtain a set of linearly independent forms, which we may employ as the n variables in terms of which Q is expressed. By Lemma 1 this change of variables leaves the characteristic invariant. Thus the problem which arises when the Z's are set equal to zero reverts to the above case where $x_1 = \cdots = x_{e-1} = 0$.

THEOREM 1. *The minimum τ for which a quadratic form Q with rank r and characteristic σ has the expansion* (2), *where the L's and M's are linear forms, is $r-\sigma$.*

We suppose that Q is written as (2), where τ is a minimum. If for some i and element k_i we have $M_i \equiv k_i L_i$, we write $k_i L_i^2$ in place of $L_i M_i$ (i not summed). Thus we can split the sum (2) into $R+S$, where

$$(5) \qquad R = \sum_{i=1}^{s} L_i M_i, \qquad S = \sum_{i=1}^{t} \nu_i N_i^2,$$

L_i being linearly independent of M_i for each i, and the N's being linear forms. The L's form a set of linearly independent linear forms, since otherwise we can write R as a sum of products of linear forms with less terms. As in §1 we write a canonical splitting of Q as $G+H$. Since Q has rank r, we may take Q to be a form in r independent variables. Since the rank of S is t, we have $t \geqq r-2s$. If $s<\sigma$, we have $s+t>r-\sigma$, whence the decomposition corresponding to the canonical splitting $G+H$ has less terms than (2). Thus $s \geqq \sigma$, and we can write $s=\sigma+\rho$ for a $\rho \geqq 0$.

We relabel the subscripts on the L's, M's, and N's if necessary so that the forms in the set Σ, where

$$\Sigma = (L_1, \cdots, L_{\sigma+\rho}, M_1, \cdots, M_\zeta, N_1, \cdots, N_\xi),$$

yield a minimal basis for the L's, M's and N's. Here $\xi \geqq r-\sigma-\rho-\zeta$. If (2) is a more economical decomposition than that which arises from $G+H$, we have $t \leqq r-\rho-2\sigma-1$. Now $t \geqq r-\sigma-\rho-\zeta$. These inequalities yield $\zeta \geqq \sigma+1$. We suppose that ζ satisfies this inequality. We let Q' designate the form

$$Q - \sum_{i=\xi+1}^{t} \nu_i N_i^2.$$

Since by Lemma 2 the characteristic changes at most by 1 under each subtraction with $\nu_i N_i^2$ (i not summed), the index α of Q' is such that

$$\alpha \leqq \sigma + t - \xi.$$

Eliminating t with the aid of an inequality relation above, we have $\alpha \leqq \zeta - 1$.

We take the linear forms in the set Σ to be the variables in terms of which the form Q' above is expressed. Setting $L_{\zeta+1}, \cdots, L_{\sigma+\rho}=0$, we obtain from Q' a form Q'' with index ζ. By Lemma 3, we have $\alpha \geqq \zeta$, giving us a contradiction. It follows that $\tau=r-\sigma$.

For the complex field the characteristic σ of Q is $[r/2]$, whereas for the real field σ is the minimum of the indices of Q and $-Q$. These results yield Corollary 1.

COROLLARY 1. *For the complex field the minimum number τ of Theorem* 1 *is $r-[r/2]$. For the real field τ is the maximum of the indices of Q and $-Q$.*

Witt proved[5] that a form is a zero form if and only if the characteristic σ of this form is greater than 0.

COROLLARY 2. *The form Q of Theorem* 1 *is a zero form if and only if $\tau \neq r$.*

By Theorem 1 the sum (2), where τ is a minimum, is a sum with the R and S of (5) satisfying $R=G$, $S=H$, the sum $G+H$ being a canonical splitting of Q.

Although addition of a term LM, L and M linear, may change the rank r of Q by 2, this is not true of the index σ and $\tau=r-\sigma$ as we shall now prove.

THEOREM 2. *Under addition of a term LM, L and M linear, to a quadratic form Q, the characteristic σ, and the minimum number τ for decompositions of type* (2), *change at most by* 1.

We write Q as a sum (2) where τ takes on the minimum value $r-\sigma$, the rank of Q being r. We let τ', r', σ' designate the analogues for $Q'=Q+LM$ of τ, r, σ for Q. Since

$$Q' = \sum_{i=1}^{\tau} L_iM_i + LM,$$

we clearly have $\tau' \leqq \tau+1$. Thus τ changes at most by 1 under the addition of LM to Q.

We suppose that L and M are linearly independent of each other and of the variables in Q so that $r'=r+2$. We write Q as $R+S$, where R and S are given by (5) with $s=\sigma$, $t=r-\sigma$, whence

$$Q \equiv \sum_{i=1}^{\sigma} L_iM_i + \sum_{i=1}^{r-\sigma} \nu_i N_i^2. \tag{6}$$

[5] See the above reference to a paper by E. Witt

The form R', where $R'=R+LM$, has index $\sigma+1$ and rank $2(\sigma+1)$, from which it follows that Q' has the canonical splitting $R'+S'$ with $S'\equiv S$. Thus Q' has index $\sigma+1$.

If L and M are taken linearly dependent, or one or both of the forms L, M are restricted to be linear forms in the variables of Q, by Lemma 3 we obtain from the form Q' of the preceding paragraph a form Q^* whose characteristic does not exceed $\sigma+1$. Thus in any case $\sigma'\leqq\sigma+1$, whence also $\sigma\leqq\sigma'+1$.

If the rank of $Q+LM$ is less than the rank of Q, Theorem 2 implies that the characteristic of $Q+LM$ does not exceed that of Q, whereas if the addition of LM to Q decreases the rank of Q by 2, this addition also decreases the characteristic of Q.

We have the following analogue of Theorem 2.

THEOREM 3. *Under addition of a term LM, L and M linear, to a real quadratic form Q, the index of Q changes at most by* 1.

We write Q as the sum $P+N$ of a positive definite form N and a negative definite form N, the rank of Q being the sum of the ranks of P and N. We suppose that Q is written in any way as a sum $P'+N'$, where P' and N' are positive definite and negative definite forms respectively. The index α of Q is the rank of P. We let β designate the rank of P', whence

$$P' = \sum_{i=1}^{\beta} p_i P_i^2$$

for linear forms $P_1, \cdots, P_\beta$. We suppose that $\alpha>\beta$. Setting $P_1=\cdots=P_\beta=0$, we have $P\not\equiv 0$, while $P+N$ is negative definite, a contradiction. Thus $\alpha\leqq\beta$.

The form LM can be written as the difference R^2-S^2, where R and S are linear forms, or one of the terms R, S is zero. The form $Q+LM$ is a sum of the positive definite form $P+R^2$, and negative definite part $N-S^2$. Since the rank of $P+R^2$ differs at most by 1 from the rank of P, the index of $Q+LM$ does not exceed $\alpha+1$. It follows that the indices of Q and $Q+LM$ differ at most by 1.

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