A CONVERGENCE THEOREM FOR CERTAIN LAGRANGE INTERPOLATION POLYNOMIALS

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In the Lagrange interpolation polynomial $L_n[f;\theta]$ where

$$L_n[f;\theta] \equiv \sum_{k=1}^{n} f(x_k) l_k[\theta],$$

(1)
$$l_k[\theta] \equiv l_k^{(n)}[\theta] \equiv l_k(x) \equiv \frac{\phi_n(x)}{\phi_n'(x_k)(x - x_k)},$$
$$\phi_n(x) \equiv \prod_{k=1}^{n} (x - x_k),$$
$$x = \cos\theta;\ -1 < x_k < 1;\ k = 1, 2, \cdots, n;\ n = 1, 2, \cdots,$$

and $f(x)$ is a continuous function defined in $(-1, 1)$, we suppose that

(2)
$$x_k \equiv x_k^{(n)} = \cos\theta_k = \cos k\pi/(n+1).$$

Then [1],[1] we have

(3)
$$\phi_n(x) = \frac{\sin(n+1)\theta}{2^n \sin\theta}, \qquad x = \cos\theta,$$
$$l_k[\theta] = \frac{(-1)^{k+1}\sin^2\theta_k \sin(n+1)\theta}{(n+1)\sin\theta(\cos\theta - \cos\theta_k)}.$$

We introduce the following notations:

(4)
$$t_n \equiv t \equiv \theta_1/2 \equiv \pi/2(n+1), \qquad M = \max_{-1\leqq x\leqq 1} |f(x)|,$$
$$S_k[\theta] \equiv \{l_k[\theta - t] + l_k[\theta + t]\}/2.$$

We shall prove the following theorem which was suggested by a similar theorem of Grünwald [2].

THEOREM. *Let $f(x)$ be a continuous function in the interval $-1 \leqq x \leqq 1$. Then*

(5)
$$\lim_{n\to\infty} (1/2)\{L_n[f;\theta - t_n] + L_n[f;\theta + t_n]\} = f(\cos\theta), \quad 0 < \theta < \pi,$$

and the convergence is uniform in the interval $0 < \alpha \leqq \theta \leqq \pi - \alpha$ (α arbi-

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[1] The numbers in brackets refer to the bibliography.

rary, fixed constant). In general, convergence does not hold for $\theta=0$ *or* $\theta=\pi$.

We shall prove first that there is a constant D for which

$$\sum_{k=1}^{n} |S_k[\theta]| < D, \qquad 0 < \alpha \leqq \theta \leqq \pi - \alpha;\ n = 1, 2, \cdots . \tag{6}$$

If $\theta \neq \theta_k \pm t$, it follows from (3) and (4), by the use of trigonometric addition formulas, that

$$\begin{aligned} S_k[\theta] &= \frac{(-1)^{k+1} \sin t \sin^2 \theta_k \cos (n+1)\theta}{4(n+1) \sin (\theta - t) \sin (\theta + t)} \\ &\cdot \frac{\cos\theta\cos\theta_k - \cos 2\theta \cos t}{\sin (\theta+\theta_k-t)/2 \sin (\theta-\theta_k-t)/2 \sin (\theta+\theta_k+t)/2 \sin (\theta-\theta_k+t)/2}. \end{aligned} \tag{7}$$

If θ is restricted to the interval $0<\alpha\leqq\theta\leqq\pi/2$, and if n is large so that $t\leqq\alpha/2$, it is easily seen (assuming $\alpha\leqq\pi/3$) that

$$\begin{aligned} &\left| \frac{\sin^2 \theta_k}{\sin (1/2)(\theta + \theta_k - t) \sin (1/2)(\theta + \theta_k + t)} \right| < 4, \\ &| S_k[\theta] | < \frac{\pi^3 C}{(n+1)^2 |\theta - \theta_k - t| \cdot |\theta - \theta_k + t|}, \\ &| \csc (\theta - \lambda) \csc (\theta + \lambda) | \leqq \frac{2}{\cos\alpha - \cos 2\alpha} \equiv C, \\ &\qquad\qquad \alpha \leqq \theta \leqq \pi/2,\ 0 \leqq \lambda \leqq \alpha/2. \end{aligned} \tag{8}$$

For a given θ, there are at most two values of k for which $|\theta-\theta_k| <\pi/(n+1)$. Since [1] $|l_k(x)| <2$ $(-1\leqq x\leqq 1;\ k=1, 2, \cdots, n;\ n=1, 2, \cdots)$, from (8) we have

$$\begin{aligned} \sum_{k=1}^{n} |S_k[\theta]| &< 4 + \sum_{1\leqq k\leqq n, |\theta-\theta_k|\geqq \pi/(n+1)} |S_k[\theta]| \\ &< 4 + \frac{\pi^3 C}{(n+1)^2} \sum_{1\leqq k\leqq n, |\theta-\theta_k|\geqq \pi/(n+1)} \frac{1}{|\theta-\theta_k-t| \cdot |\theta-\theta_k+t|} \\ &< 4 + \frac{2\pi^3 C}{(n+1)^2} \sum_{l=1}^{n} \left[\frac{2(n+1)}{\pi}\right]^2 \frac{1}{(2l-1)^2} \\ &< 4 + 8\pi C \sum_{m=1}^{\infty} \frac{1}{m^2} \equiv D', \qquad 0<\alpha\leqq\theta\leqq\pi/2;\ n\geqq n_0(\alpha), \end{aligned} \tag{9}$$

because in the summation

$$|\theta - \theta_k \pm t| \geqq |\theta - \theta_k| - t \geqq l\pi/(n+1) - \pi/2(n+1), \qquad l \geqq 1.$$

By continuity, (9) holds for all θ in $0<\alpha\leqq\theta\leqq\pi/2$. Since (9) is valid for n sufficiently large, there exists a D for which (6) is valid for all n if $0<\alpha\leqq\theta\leqq\pi/2$.

Since $\theta_k=\pi-\theta_{n-k+1}$, it is found that

$$l_k[\theta - t] = l_{n-k+1}[\theta' + t] \quad \text{and} \quad l_k[\theta + t] = l_{n-k+1}[\theta' - t]$$

where $\theta'=\pi-\theta$. It follows that

$$S_k[\theta] = S_{n-k+1}[\theta'], \qquad \sum_{k=1}^{n} |S_k[\theta']| < D, \qquad \pi/2 \leqq \theta' \leqq \pi - \alpha.$$

This completes the proof of (6).

From (8), if δ (>0) is fixed and if n is sufficiently large (so that $t<\delta/2$), it is seen that

$$\begin{aligned}(10)\qquad \sum_{1\leqq k\leqq n,\, |\theta-\theta_k|>\delta} |S_k[\theta]| &< \frac{\pi^3 C}{(n+1)^2}\sum_{k=1}^{n}\frac{1}{(\delta-t)^2} \\ &< \frac{4\pi^3 C n}{\delta^2(n+1)^2} = O\left(\frac{1}{n}\right).\end{aligned}$$

We are now ready to prove the main part of the theorem. Let θ be fixed ($0<\alpha\leqq\theta\leqq\pi-\alpha$) and $\epsilon>0$. It is well known that

$$\sum_{k=1}^{n} l_k(x) \equiv 1, \qquad \sum_{k=1}^{n} S_k[\theta] \equiv 1.$$

Since $f(x)$ is continuous, there exists a $\delta>0$ such that

$$|f(\cos\theta) - f(\cos\theta_k)| < \epsilon \quad \text{provided} \quad |\theta - \theta_k| \leqq \delta.$$

Let

$$\begin{aligned}\Delta &= (1/2)\{L_n[f;\theta - t] + L_n[f;\theta + t]\} - f(\cos\theta) \\ &= \sum_{k=1}^{n}\{f(\cos\theta_k) - f(\cos\theta)\}S_k[\theta] \\ &= \sum_{1\leqq k\leqq n,\,|\theta-\theta_k|\leqq\delta}\{f(\cos\theta_k) - f(\cos\theta)\}S_k[\theta] \\ &\quad + \sum_{1\leqq k\leqq n,\,|\theta-\theta_k|>\delta}\{f(\cos\theta_k) - f(\cos\theta)\}S_k[\theta].\end{aligned}$$

Then, by the use of (6) and (10), for sufficiently large n, we find that

$$|\Delta| < \epsilon \sum_{1\leqq k\leqq n, |\theta-\theta_k|\leqq\delta} |S_k[\theta]| + 2M \sum_{1\leqq k\leqq n, |\theta-\theta_k|>\delta} |S_k[\theta]|$$

$$< \epsilon D + 2MO\left(\frac{1}{n}\right) < (D+1)\epsilon, \qquad n > N.$$

As in Rogosinki's theorem for Fourier series, the theorem may be easily extended so that convergence holds at any point x ($\neq \pm 1$) of continuity of a bounded function $f(x)$, convergence being uniform in any closed interval of continuity (excluding $x = \pm 1$). In addition, t_n may be replaced by $p\pi/2(n+1)$ where p is any fixed odd integer.

Since $L_n[f; -t] = L_n[f; t]$, the theorem would involve (for $\theta = 0$) the convergence of $L_n[f; t]$ to $f(\cos 0)$. This convergence does not hold for all continuous $f(x)$ because

$$\lambda_n \equiv \sum_{k=1}^{n} |l_k(1)| = n,$$

and, according to H. Hahn [3], a necessary and sufficient condition for $L_n[f; 0]$ to converge to $f(\cos 0)$ for all continuous $f(x)$ is that λ_n be bounded for all n. We give an example (compare [4] and [5]) of a continuous function for which convergence does not occur at $x = 1$.

Let $f_n(x)$ be defined for each n ($n = 1, 2, \cdots$) as follows:

$$(11)\quad f_n(x) = \begin{cases} (-1)^{n-1}, & -1 \leqq x < x_n, \\ (-1)^k\left[1 - \dfrac{2(x - x_{k+1})}{x_k - x_{k+1}}\right], & x_{k+1} \leqq x \leqq x_k;\ k = 1, 2, \cdots, n-1, \\ 1, & x_1 < x \leqq 1. \end{cases}$$

Now,

$$L_n[f_n; t] = -\sum_{k=1}^{n} (-1)^k l_k[t]$$

$$= \frac{2}{(n+1)\sin t} \sum_{k=1}^{n} \frac{\sin^2 kt \cos^2 kt}{\sin (k+1/2)t \sin (k-1/2)t} \cdot$$

Since

$$\frac{\sin kt}{\sin (k+1/2)t} > \frac{2}{3}, \qquad k = 1, 2, \cdots, n,$$

we have

$$(12)\quad \begin{aligned} |L_n[f_n; t]| &> \frac{4}{3(n+1)\sin t} \sum_{k=1}^{n} \cos^2 kt \\ &> \frac{2}{3(n+1)\sin t} \sum_{k=1}^{n} (1 + \cos\theta_k) > \frac{4n}{3\pi} \cdot \end{aligned}$$

By the Weierstrass approximation theorem, we may approximate $f_n(x)$ by a polynomial $g_n(x)$ of degree $d(n)$ such that in $(-1, 1)$,

$$(13) \qquad | g_n(x) | \leqq 3/2, \qquad | L_n[g_n; t] | > 2n/3\pi.$$

This is possible, in view of (12), because

$$| L_n[f_n; t] - L_n[g_n; t] | = \left| \sum_{k=1}^{n} [f_n(x_k) - g_n(x_k)] l_k[t] \right| \leqq \epsilon'$$

provided

$$| f_n(x) - g_n(x) | \leqq \left(\epsilon' \Big/ \sum_{k=1}^{n} | l_k[t] | \right), \qquad -1 \leqq x \leqq 1.$$

Let

$$(14) \qquad g(x) \equiv \sum_{i=1}^{\infty} c_i g_{n_i}(x), \qquad -1 \leqq x \leqq 1,$$

where $c_1 = n_1 = 1$, and

$$(15) \qquad c_{i+1} = \min\left\{\frac{c_i}{4}, \frac{1}{T_i}\right\}, \qquad T_i \equiv \sum_{k=1}^{n_i} | l_k^{(n_i)}[t_{n_i}] |, \qquad i = 1, 2, \cdots;$$

n_i is the smallest integer satisfying the conditions

$$(16) \qquad \begin{aligned} &\text{(a)} \quad n_i \geqq d(n_{i-1}) + 1, \\ &\text{(b)} \quad | c_i L_{n_i}[g_{n_i}; t_{n_i}] - 8 | > 4^i, \qquad i = 2, 3, \cdots. \end{aligned}$$

Condition (b) is possible because of (13). From (15), it follows that

$$(17) \qquad c_{i+1} \leqq 1/4^i, \qquad i = 0, 1, 2, \cdots,$$

and the series for $g(x)$ converges uniformly so that $g(x)$ is continuous and $|g(x)| \leqq 2$ in $(-1, 1)$. Let $m = n_r$ where r $(\neq 1)$ is a positive integer. Using (15) and (17), we have

$$(18) \qquad \begin{aligned} \left| \sum_{i=r+1}^{\infty} c_i L_m[g_{n_i}; t_m] \right| &\leqq \sum_{i=r+1}^{\infty} c_i \frac{3}{2} \sum_{k=1}^{m} | l_k^{(m)}[t_m] | \leqq 2, \\ \left| \sum_{i=r}^{\infty} c_i g_{n_i}(\cos t_m) \right| &\leqq 2. \end{aligned}$$

If $w(x)$ is any polynomial of degree less than m in $x = \cos \theta$, then $L_m[w; \theta] \equiv w(x)$. Since

$$\begin{aligned}\sum_{i=1}^{\infty} c_i L_m[g_{n_i}; t_m] &= c_r L_m[g_m; t_m] + \sum_{i=1}^{r-1} c_i L_m[g_{n_i}; t_m] + \sum_{i=r+1}^{\infty} c_i L_m[g_{n_i}; t_m] \\ &= c_r L_m[g_m; t_m] + \sum_{i=1}^{r-1} c_i g_{n_i}(\cos t_m) + \sum_{i=r+1}^{\infty} c_i L_m[g_{n_i}; t_m],\end{aligned}$$

we find from (16) and (18) that

$$\begin{aligned}|L_m[g; t_m] - g(1)| &= \left| \{L_m[g; t_m] - g(\cos t_m)\} + \{g(\cos t_m) - g(1)\} \right| \\ &\geqq \left| \sum_{i=1}^{\infty} c_i L_m[g_{n_i}; t_m] - g(\cos t_m) \right| - 4 \\ &\geqq \left| c_r L_m[g_m; t_m] + \sum_{i=r+1}^{\infty} c_i L_m[g_{n_i}; t_m] - \sum_{i=r}^{\infty} c_i g_{n_i}(\cos t_m) \right| - 4 \\ &> 4^r, \qquad r = 2, 3, \cdots .\end{aligned}$$

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