ON AN INEQUALITY OF SEIDEL AND WALSH

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Introduction. In a recent paper[1] Seidel and Walsh introduced the following concepts.

Let R be a Riemann surface (configuration) lying over the w-plane, and let C_p be a simply-connected region of R having the following properties:

(a) C_p contains precisely p points (counted according to branch-point multiplicity) lying over some point of the w-plane.

(b) C_p lies over the circle $|w-w_0|<r$, and the boundary of C_p lies over the circumference $|w-w_0|=r$.

It follows that C_p contains precisely p points lying over every point of $|w-w_0|<r$, and in particular, p points $\bar{w}_i$ lying over w_0. Seidel and Walsh name such a region a *p-sheeted circle with centers $\bar{w}_i$ and radius r.* Given a point $\bar{w}_0$ of R, let r_p be the radius of the largest p-sheeted circle in R with center $\bar{w}_0$; if none exists, let $r_p=0$. We then define the *radius of p-valence of R at $\bar{w}_0$*, $D_p(\bar{w}_0)$, as the maximum of the r_n for $n\leqq p$.

Let $w=f(z)=a_1z+\cdots+a_pz^p+a_{p+1}z^{p+1}+\cdots$ be analytic in the unit circle $|z|<1$ with $|f(z)|<M$, and let the Riemann surface R be the image of $|z|<1$ under $w=f(z)$. Let $\bar{w}_0$ be the image of $z=0$; $\bar{w}_0$ lies over $w=0$. Seidel and Walsh establish the following relation between the first p coefficients of $f(z)$ and the radius of p-valence, $D_p(\bar{w}_0)$, of R at $\bar{w}_0$.

There exist two constants, λ_p depending only on p, and Λ_p depending on p and M, such that

$$\lambda_p D_p(\bar{w}_0) \leqq \sum_{n=1}^{p} |a_n| \leqq \Lambda_p [D_p(\bar{w}_0)]^{2^{-p}}. \tag{1}$$

Seidel and Walsh find for Λ_p the value

$$\Lambda_p = 24pM^r, \qquad r = 1 - 2^{-p}.$$

In this note we prove the following two statements concerning the inequalities (1).

A. *The exponent 2^{-p} may be replaced by $1/(p+1)$ and this exponent is the best possible (for $D_p\to 0$).*

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[1] W. Seidel and J. L. Walsh, *On the derivatives of functions analytic in the unit circle and their radii of univalence and of p-valence*, Transactions of this Society, vol. 52 (1942), pp. 129–216.

We show, in fact, that there is a constant K_p depending only on p such that

$$\sum_{n=1}^{p} | a_n | \leqq K_p M^{p/(p+1)} [D_p(\bar{w}_0)]^{1/(p+1)}, \tag{2}$$

and that there is a class of functions actually attaining the bound M for which

$$\sum_{n=1}^{p} | a_n | \geqq M^{p/(p+1)} [\dot{D}_p(\bar{w}_0)]^{1/(p+1)}. \tag{3}$$

B. *The coefficient* λ_p *may be replaced by* 1, *and this value of the constant is the best possible.*

All proofs are based on the theorem of Rouché. We shall always suppose, without explicit statement, that z is confined to the unit circle $|z| < 1$.

Proof of (2). To prove (2) we assume $a_p \neq 0$ and let

$$P(z) = \sum_{n=1}^{p} a_n z^n = a_p \prod_{n=1}^{p} (z - b_n), \tag{4}$$

from which we have

$$| P(z) | \leqq \sum_{n=1}^{p} | a_n | = C. \tag{5}$$

We observe that $|a_n| \leqq M$, so that $|P(z)| < pM$, and therefore by the Schwarz lemma,

$$| f(z) - P(z) | < (p + 1)M | z |^{p+1}. \tag{6}$$

Now let

$$\begin{aligned} P_1(r) &= | a_p | \prod_{n=1}^{p} (r - | b_n |) = c_1 r + \cdots + c_p r^p, \\ C' &= \sum_{n=1}^{p} | c_n | = | a_p | \prod_{n=1}^{p} (1 + | b_n |). \end{aligned} \tag{7}$$

Then by (4) and (7), $|P(re^{i\theta})| \geqq |P_1(r)|$. Also, since c_n is the same polynomial in the $|b_j|$ as a_n is in the b_j, and since only plus signs (or only minus signs) occur, we have $|c_n| \geqq |a_n|$, and $C' \geqq C$. Together with (7) this gives

$$| P(re^{i\theta}) | \geqq C \left| \prod_{n=1}^{p} \frac{r - | b_n |}{1 + | b_n |} \right|. \tag{8}$$

Note that if $|b_i|>1$ we only strengthen the inequality (8) if we replace $|b_i|$ by 1 in the quotients occurring there. But then (8) implies that

$$(9)\qquad |P(re^{i\theta})| \geqq C2^{-p}\prod_{n=1}^{p}|r-b_n'|,$$

where $b_n'=|b_n|$ if $|b_n|\leqq 1$, and $b_n'=1$ otherwise. Now choose any positive real number $x_0<1$. In the interval $(0, x_0)$ there is at least one point r_1 such that for $i\leqq p$,

$$|r_1-b_i'| \geqq x_0/2^p,$$

and therefore by (9),

$$(10)\qquad |P(r_1e^{i\theta})| \geqq Cx_0^p(4p)^{-p},$$

and by (6) and (10)

$$|P(r_1e^{i\theta})| - |f(r_1e^{i\theta}) - P(r_1e^{i\theta})| \geqq Cx_0^p(4p)^{-p} - x_0^{p+1}(p+1)M.$$

This expression is maximum for $x_0=Cp/(4p)^p(p+1)^2M$, its maximum value being C^{p+1}/M^pK^{p+1}, where $K=(p+1)((p+1)/p)^{p/(p+1)}(4p)^p$.

The polynomial $P(z)$ has n zeros in $|z|\leqq r_1$ with $n\leqq p$. Therefore by Rouché's theorem, $f(z)$ assumes in $|z|\leqq r_1$ every value in $|w| \leqq C^{p+1}/K^{p+1}M^p$ precisely n times. Therefore $D_p(0)\geqq C^{p+1}/K^{p+1}M^p$; that is,

$$\sum_{n=1}^{p}|a_n| \leqq KM^{p/(p+1)}[D_p(0)]^{1/(p+1)}$$

as was to be proved.

In the above proof we have assumed that $a_p\neq 0$. If $a_p=0$ the proof is valid for some $n\leqq p$, and p can be reintroduced in the later stages, say in equation (8), where the product can be taken over p factors if some of the $|b_n|$ are allowed to vanish.

Proof of (3). We prove (3) by considering the class of functions

$$(11)\qquad f(z) \equiv Mz^p\frac{a-Mz}{M-az}$$

$$(12)\qquad \equiv az^p\frac{1-(Mz/a)}{1-(az/M)}$$

where $a<M$. These functions have been studied by the author in a

previous note.[2] From (11) it is clear that $|f(z)| \leqq M$ in $|z| \leqq 1$, and that the bound is attained for $z=1$. It is also evident from (11) that the minimum of $|f(z)|$ on $|z| = r < a/M$ occurs at $z=r$. Thus $|f(z)| \geqq f(r)$ for $|z| = r < a/M$. By Rouché's theorem $f(z)$ covers the circle $|w| < f(r)$ precisely p times in $|z| < r$, since $f(z)=0$ has p roots in this circle. By Rolle's theorem $f'(z)$ vanishes for some r_0 between $r=0$ and $r=a/M$. Then every value of $|w| \leqq f(r_0)$ except $w=f(r_0)$ is assumed precisely p times in $|z| \leqq r_0$, and $w=f(r_0)$ is assumed $p+1$ times. Therefore $D_p(0)=f(r_0)$. But by (12)

$$D_p(0) = f(r_0) < a(r_0)^p < a(a/M)^p = a^{p+1}/M^p. \tag{13}$$

Also by (12), the first term in the expansion of $f(z)$ is az^p so that the sum of the moduli of the first p coefficients is a. Thus by (13)

$$\sum_{n=1}^{p} |a_n| = a > M^{p/(p+1)}[D_p(0)]^{1/(p+1)}$$

which is the desired inequality (3).

The coefficient λ_p. We have left to prove that λ_p can be taken as 1; in other words, that

$$D_p(0) \leqq \sum_{n=1}^{p} |a_n| = C.$$

As before, let $P(z) = a_1z + \cdots + a_pz^p$, and suppose that $D_p(0) > C$. Then $|f(z)| = C$ defines an analytic Jordan curve Γ in $|z| < 1$ and Γ contains the origin in its interior. On Γ we have $|P(z)| < C = |f(z)|$. Therefore by Rouché's theorem $f(z) - P(z)$ and $f(z)$ have the same number of zeros interior to Γ. But $f(z)$ has at most p zeros inside Γ since the image of the interior of Γ under $f(z)$ is an n-sheeted circle with $n \leqq p$. And $f(z) - P(z) \equiv a_{p+1}z^{p+1} + \cdots$ has at least $p+1$ zeros interior to Γ. By this contradiction we see that $D_p(0) \leqq C$ as we wished to prove.

This result is trivially the best possible, for if $f(z) \equiv az^p$, then $D_p(0) = a = C = M$.

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[2] L. H. Loomis, *The radius and modulus of n-valence for analytic functions whose first n−1 derivatives vanish at a point*, this Bulletin, vol. 46 (1940), pp. 406–501.

Two problems were considered in this note. One was the problem of the present paper for the class of functions whose first non-vanishing coefficient is a_p. The second was the problem of determining for this restricted class of functions the largest circle $|z| < r$ (the radius depending on p, M and $|a_p|$) in which all the functions are at most p-valent. The present note does not treat the corresponding problem for the more general class of functions considered here.