BOOK REVIEWS

What is Mathematics? By Richard Courant and Herbert Robbins. London, New York and Toronto, Oxford University Press, 1941. 19+521 pp. $5.00.

Professor Courant is too expert a hand at mathematics to attempt a formal definition of the subject. Instead, he and his collaborator offer "an elementary approach to ideas and methods," from which the reader who persists to the end may draw his own conclusions. It will be interesting to American teachers and students to forecast what a few of those conclusions may be.

First, as to the material presented. Roughly, the topics treated cover the usual superior undergraduate course in mathematics through the calculus, with glimpses of such things as Riemannian geometry and Plateau's problem beyond the customary curriculum. It is a misapprehension, under which some authors not acquainted at first hand with mathematics as taught in the better colleges labor, that mathematics for the American undergraduate ends with the calculus, or at farthest with an introduction to differential equations. It does not; quite detailed courses in the theory of algebraic equations, the theory of numbers, projective geometry, both synthetic and analytic, modern plane elementary geometry, also occasionally non-Euclidean geometry, are offered and taken; and in some of the undergraduate schools of the larger universities a course in the theory of functions of a complex variable is a commonplace, as also is a thorough introduction to mechanics. So the average student or graduate of the better college courses in mathematics will not be wholly unprepared to appreciate the numerous illuminating sidelights which this book offers on what he already knows; nor will he be entirely blind to the attractions of things which he may see here for the first time. All this sums up to the opinion that the book is one for inspirational collateral reading, rather than a detailed manual for the mastery of any one of the topics it treats. The hypothetical layman who remembers a little of what he learned in college ten to fifty years ago will find the book both stimulating and demanding.

In the method of presentation, at least, there is something new and interesting for nearly every topic, if only a brief note calling attention to significant progress made within the past decade, or an excellent drawing making textual comment all but superfluous. A sound pedagogical strategy ensnares the reader at the outset (pp. 1–51) in the serenely useless properties of the common whole numbers, catching his attention with simple but singularly recondite problems, solved and unsolved, while painlessly indoctrinating him with such useful things as the binomial theorem and geometrical progressions. The number system of analysis is then developed, culminating (p. 72) in the Dedekind cut as an alternative to the method of nested intervals previously developed in the text. After a brief excursus on the basic principle of analytic geometry, the discussion of the continuum is resumed in "the mathematical analysis of infinity," where the paradoxes and the present disorderly state of the foundations of mathematics are fairly and fearlessly discussed. Teachers who still believe such things too scandalous to be mentioned in the presence of the young, may find here just the temperate account they have been seeking. After all, college students are old enough not to be unduly shocked by the discovery that the Queen of the Sciences is no better than she should be.

So far, mathematics has appeared as a rather gentlemanly pursuit, interesting enough to those who like the sort of thing mathematics is (even if they do not know, and are not told, what it is), but essentially useless. Indeed, the emphasis throughout the book is on principles rather than on trite imitations of the genuinely practical applications of mathematics such as a modern engineer or theoretical physicist must know. In this connection, differentials (p. 435) as misused to great advantage by physicists, engineers and other practical men who get tangible things done, come in for some bad moments, and the hapless engineer who imagines his differentials as little bits of reality is left without a d to his x. Cannot something be done to span the chasm dividing unsound mathematics that helps to produce things that make money, from sound mathematics that barely pays a beggarly living wage to instructors in our colleges and universities? Rebuking the engineers and physicists for their differential and infinitesimal sins may relieve a pure mathematician's outraged emotions, but it will pay nobody's grocery bill, unless the mathematician is lucky enough to make a smash-hit with a best-selling text, as we hope this one will be.

Off the familiar highway are such things as items from topology in Chapters IV, V, where, incidentally, the Jordan curve theorem is described, perhaps to disabuse anyone who still takes too seriously the dictum of Gauss that "mathematics is the science of the eye." The balance between intuition and logical rigor, here as elsewhere, is justly preserved, and bigots for either intuition or logic as the sole guarantor of mathematical truth will find but little support for their exclusive dogmas. Interesting details that will be new to a majority of the readers to whom the book is addressed are Brouwer's fixed point theorem, the genus of a surface, Klein's tantalizing bottle (the drunkard's deliverance), and the five color problem. To an algebraist it may seem somewhat antiquated to present only the analytic version of the fundamental theorem of algebra. Topology appears again in connection with minimax points; and generally the discussion of maxima and minima is as unhackneyed as anything in the book. Even Dirichlet's principle is described, possibly as a danger signal that intuition alone is not always enough, even in applied mathematics. Surely this must be an innovation in a college text. As the senior author has worked on Plateau's problem, he indulges his fancy for soap films in an interesting section suggesting that what mathematics slaves through pages of intricate analysis to accomplish, is sometimes done effortlessly by nature in a single gesture.

Enough has been sampled to indicate that here is an unusually rich text, full of suggestions for further reading, and replete with interesting and beautiful things not found in other books at the same level. The authors have reworked some of the best from each of many sources, due acknowledgement being made in the text and in the bibliography; the welding of it all into a homogeneous unity is exclusively their own. Sometimes the style is reminiscent of Klein; at other times Hilbert is plainly visible in the background; but all in all the work is an expression of independent individuality.

There is one aspect of the book which may be the most significant of all for young students looking forward to a career in mathematics. The emphasis of the recent past in American mathematics has been heavily on the side of axiomatics and abstraction; this book reminds us that intuition also has its part in mathematical creation.

Within the last decade several texts aiming to break away from the deadening drill in formal manipulation and theorems that are now only museum pieces have been written for American students. Some have favored mathematics as a logical discipline; others have presented it as a fine art; and yet others have described the power of mathematics in science and daily affairs. Each has had its definite function to perform, and all have contributed to a long overdue enlightenment in the teaching of mathematics at the college level. The present book differs in many respects from its predecessors; the points of difference make it a most welcome addition to the library of modernized teaching in mathematics.

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