is a regular Hausdorff mean. By (4.1) we see that $[H, q_n]\supset(C, 1)$. It is easy to show that $(C, 1)\supset [H, q_n]$, thereby proving that $[H, q_n] \approx (C, 1)$. If $c_n = p_n = 1/(n+1)$, then

$$q_n = \frac{1 + 2^{-1} + 3^{-1} + \cdots + n^{-1}}{n+1}$$

and $[H, q_n]$ is a regular Hausdorff mean. This mean does not include $(C, 1)$ inasmuch as q_n: $(n+1)^{-1}$ is unbounded.

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WHIRL-SIMILITUDES, EUCLIDEAN KINEMATICS, AND NON-EUCLIDEAN GEOMETRY

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1. **Introduction.** The geometry of whirls and whirl-motions in the plane had its origin in a paper by E. Kasner [6],[1] was subsequently developed in a series of papers by Kasner and DeCicco [**3, 7, 8, 9**], adapted to the sphere by Strubecker [**10**], and to 3-space by Feld [**4**]. In this paper we shall, by adjoining three involutory transformations, extend Kasner's whirl-motion group G_6 to a mixed group Γ_6—the *complete whirl-motion group*—composed of eight mutually exclusive, six-parameter families; these families will in turn be extended to seven-parameter families comprising the mixed group Γ_7—the *complete whirl-similitude group*. The principal results obtained are the extension of Kasner's G_6 and two representations of Γ_7: a *kinematic representation on the plane*, §6, and a *representation in quasi-elliptic* 3-*space*, §7.

2. **Slides, turns, and whirls.** Let the point of an oriented lineal element E have the rectangular coordinates x, y, and let the inclination of E to the x-axis be the angle θ, $0 \leqq \theta < 2\pi$. Let $z = x + iy$, $\bar{z} = x - iy$, $\zeta = e^{i\theta}$. We shall call z, ζ the *element coordinates* of E (x, y, θ), which, henceforth, shall be represented by the symbol (z, ζ).

DEFINITIONS. *A slide S_s is a lineal element transformation that translates the point of each element along its line the same distance s.*

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[1] The numbers in brackets refer to the bibliography at the end of the paper.

A turn T_α *is a lineal element transformation that rotates the line of every lineal element around its point through the same angle* arg α.

A direct whirl[2] *is a product* $T_\alpha S_s T_\beta$.

From these definitions it follows that the slide S_s, $(z, \zeta)\to(z^*, \zeta^*)$, is given by the equations $z^*=z+s\zeta$, $\zeta^*=\zeta$, $(s=\bar{s})$; the turn T_α is given by the equations: $z^*=z$, $\zeta^*=\alpha\zeta$, $(|\alpha|=1)$; and the direct whirl $T_\alpha S_s T_\beta \equiv W^+_{c,\gamma}$ is given by $z^*=z+c\zeta$, $\zeta^*=\gamma\zeta$, where $c=\alpha s$, $\gamma=\alpha\beta$, $|\gamma|=1$.

Evidently $W^+_{a,\alpha}W^+_{b,\beta}=W^+_{c,\gamma}$ where $c=a+\alpha b$, and $\gamma=\alpha\beta$. The direct whirls form a three-parameter group $\mathfrak{W}^+_3$. The direct whirls $W^+_{c,\gamma}$ such that $c+\bar{c}=0$ and $\gamma=\pm 1$ are dilatations.

3. **Flat fields and turbines.** A *flat field* has been defined by Kasner as a totality of ∞^2 oriented lineal elements that lie on the ∞^1 cycles (oriented circles) containing a given element called the center of the flat field. If s, σ are the coordinates of the center of the flat field F, the equation of F in current lineal element coordinates z, ζ is

$$z - s = (\bar{z} - \bar{s})\sigma\zeta. \tag{3.1}$$

Inasmuch as the parameters s, σ completely characterize F, we shall call them the coordinates of F, and use the symbol $\{s, \sigma\}$ to represent the flat field whose center is (s, σ) To distinguish the coordinates of a flat field from those of a lineal element, the former shall be referred to as *dual coordinates* and the latter as *element coordinates.* A flat field $\{s, \sigma\}$ and a lineal element (z, ζ) shall be said to be *incident* if their respective coordinates satisfy (3.1). Evidently flat fields and lineal elements can be regarded as duals of one another, and (3.1), accordingly, can be interpreted both as the equation of $\{s, \sigma\}$ in element coordinates and as the equation of (z, ζ) in dual coordinates.

Direct whirls convert flat fields into flat fields. Let $W^+_{a,\alpha}$ convert $\{s, \sigma\}$ into $\{s^*, \sigma^*\}$; then we obtain for the equations of $W^+_{a,\alpha}$ expressed in dual coordinates:

$$s^* = s - \bar{a}\sigma, \qquad \sigma^* = \bar{\alpha}\sigma. \tag{3.2}$$

Two flat fields $\{a, \alpha\}$ and $\{b, \beta\}$, $\alpha\neq\beta$, have in common ∞^1 lineal elements (z, ζ) whose coordinates satisfy the equation

$$z - l = r\zeta, \tag{3.3}$$

where $l=(a\beta-b\alpha)/(\beta-\alpha)$, $r=-(\bar{a}-\bar{b})/(\bar{\alpha}-\bar{\beta})$. The points of the

[2] The term *direct whirl* is used in place of Kasner's term *whirl* to distinguish between his and another kind of whirl which will be introduced below.

lineal elements (z, ζ) on the locus (3.3) lie on the circle $(z-l)(\bar{z}-\bar{l})=r\bar{r}$ —the *outer circle* of the locus—and their oriented lines envelope a concentric *inner cycle*. Such a locus of lineal elements has been named a *turbine* by Kasner [6]. Evidently the turbine

$$z - l = - \bar{r}\zeta \tag{3.4}$$

has the same outer circle and inner cycle as (3.3). Turbines (3.3) and (3.4) shall be called *conjugate*. A necessary and sufficient condition that the turbine (3.3) be self-conjugate is that $r+\bar{r}=0$; self-conjugate turbines, and only such turbines, are cycles.

By duality, two lineal elements (a, α) and (b, β), $\alpha \neq \beta$, have in common ∞^1 flat fields $\{s, \sigma\}$ whose coordinates satisfy the equation

$$s - \frac{a\beta - b\alpha}{\beta - \alpha} = - \frac{\bar{a} - \bar{b}}{\bar{\alpha} - \bar{\beta}}\sigma. \tag{3.5}$$

This equation represents the turbine determined by (a, α) and (b, β), $\alpha \neq \beta$, as a locus of flat fields.

Since there exists a (1, 1) correspondence between turbines and the ordered pairs of complex parameters l, r, we shall let the symbol $[l, r]$ represent the turbine whose equation in element coordinates is (3.3); l and r shall be called the *turbine coordinates* of $[l, r]$. The equation of $[l, r]$ in the dual coordinates s, σ is

$$s - l = - \bar{r}\sigma. \tag{3.6}$$

The turbine $[l, r]$ and the lineal element (z, ζ) shall be said to be *incident* if their coordinates satisfy the equation (3.3); likewise $[l, r]$ and $\{s, \sigma\}$ shall be said to be incident if their coordinates satisfy (3.6).

4. **The complete group of whirls $\mathfrak{W}_3$.** Let $\mathfrak{J}_1$ represent the involutory lineal element transformation $(z, \zeta)\rightarrow(z^*, \zeta^*)$, given by the equations

$$z^* = \bar{z}, \qquad \zeta^* = \bar{\zeta}. \tag{4.1}$$

The transformations $W^-_{a,\alpha} \equiv \mathfrak{J}_1 W^+_{a,\alpha}$ shall be called *opposite whirls*. Evidently $W^+_{a,\alpha}\mathfrak{J}_1 = \mathfrak{J}_1 W^+_{\bar{a},\bar{\alpha}}$. Opposite whirls convert flat fields into flat fields and turbines into turbines. In dual coordinates, s, σ, transformation $\mathfrak{J}_1$ assumes the form

$$s^* = \bar{s}, \qquad \sigma^* = \bar{\sigma} \tag{4.2}$$

and in turbine coordinates, l, r, the form

$$l^* = \bar{l}, \qquad r^* = \bar{r}. \tag{4.3}$$

The direct and opposite whirls constitute a mixed three-parameter group of which $\mathfrak{W}_3^+$ is an invariant subgroup.

Let $\mathfrak{J}_2$ designate the transformation $(z, \zeta)\rightarrow\{s^*, \sigma^*\}$ given by the equations

$$s^* = z, \qquad \sigma^* = \zeta. \tag{4.4}$$

It is easy to ascertain that the ∞^2 lineal elements (z, ζ) incident to a given flat field $\{s, \sigma\}$ are converted by $\mathfrak{J}_2$ into the ∞^2 flat fields $\{s^*, \sigma^*\}$ incident to a unique lineal element (z^*, ζ^*); the latter shall, accordingly, be regarded as the image of $\{s, \sigma\}$ produced by $\mathfrak{J}_2$. We find that $z^*=s$, $\zeta^*=\sigma$; $\mathfrak{J}_2$ is consequently an involutory transformation. The ∞^3 transformations $C^+_{a,\alpha}\equiv\mathfrak{J}_2W^+_{a,\alpha}$ shall be called *direct correlations*, and the ∞^3 transformations $C^-_{a,\alpha}\equiv\mathfrak{J}_2W^-_{a,\alpha}$ shall be called *opposite correlations*. The direct and the opposite correlations transform turbines into turbines. In turbine coordinates l, r the involution $\mathfrak{J}_2$ assumes the form

$$l^* = l, \qquad r^* = -\bar{r}. \tag{4.5}$$

Consequently, $\mathfrak{J}_2$ *converts turbines into their conjugates.*

The four continuous, mutually exclusive, three-parameter families: $\mathfrak{W}_3^+$, $\mathfrak{W}_3^-$ (opposite whirls), $\mathfrak{C}_3^+$ (direct correlations), and $\mathfrak{C}_3^-$ (opposite correlations) constitute a mixed group—namely, the *complete group of whirls* $\mathfrak{W}_3$.

5. **Whirl-motions and whirl-similitudes.** Let the euclidean point displacement

$$z^* = \alpha z + a, \qquad |\alpha| = 1,$$

be extended to represent a displacement of lineal elements, as follows

$$z^* = \alpha z + a, \qquad \zeta^* = \alpha\zeta, \qquad |\alpha| = 1. \tag{5.1}$$

Let $D^+_{a,\alpha}$ denote the displacement of lineal elements given by (5.1), and let $\mathfrak{D}_3^+$ denote the group of such displacements. Furthermore, let $D^-_{a,\alpha}\equiv\mathfrak{J}_1D^+_{a,\alpha}$ and $\mathfrak{D}_3^-\equiv\mathfrak{J}_1\mathfrak{D}_3^+$; $\mathfrak{D}_3^-$ evidently denotes the family of euclidean symmetries operating on lineal elements. The group of euclidean motions $\mathfrak{D}_3$ $(=\mathfrak{D}_3^+ + \mathfrak{D}_3^-)$ converts flat fields into flat fields and turbines into turbines. The product of a direct whirl by a displacement is commutative. Let such a product be called a *direct whirl-motion*. The direct whirl-motions constitute a continuous, six-parameter group which shall be denoted by $\mathfrak{G}_6^1$.

Let

$$\mathfrak{G}_6^2 \equiv \mathfrak{W}_3^+\mathfrak{D}_3^-, \qquad \mathfrak{G}_6^3 \equiv \mathfrak{C}_3^+\mathfrak{D}_3^+, \qquad \mathfrak{G}_6^4 \equiv \mathfrak{C}_3^-\mathfrak{D}_3^+.$$

Evidently $\mathfrak{G}_6^2=\mathfrak{J}_1\mathfrak{G}_6^1$; $\mathfrak{G}_6^3=\mathfrak{J}_2\mathfrak{G}_6^1$; $\mathfrak{G}_6^4=\mathfrak{J}_1\mathfrak{G}_6^3$. The four families $\mathfrak{G}_6^i$ are mutually exclusive and constitute a mixed six-parameter group $\mathfrak{G}_6$, which shall be called the *group of proper whirl-motions*. The two families $\mathfrak{G}_6^1$, $\mathfrak{G}_6^2$ form a mixed invariant subgroup of $\mathfrak{G}_6$.

Let $\mathfrak{J}_3$ denote the involutory lineal element transformation

$$z^* = z\bar{\zeta}, \qquad \zeta^* = \bar{\zeta}. \tag{5.2}$$

Evidently $\mathfrak{J}_3$ transforms flat fields into flat fields and turbines into turbines. In dual coordinates $\mathfrak{J}_3$ is given by the equations

$$s^* = -\bar{s}\sigma, \qquad \sigma^* = \sigma, \tag{5.3}$$

and in turbine coordinates l, r, by the equations

$$l^* = r, \qquad r^* = l. \tag{5.4}$$

Let $\mathfrak{H}_6^i\equiv\mathfrak{J}_3\mathfrak{G}_6^i$, $i=1, 2, 3, 4$. The four, mutually exclusive, continuous families of turbine-preserving transformations $\mathfrak{H}_6^i$ comprise the *improper whirl-motions*. The totality of proper and improper whirl-motions constitute a mixed group composed of eight continuous families, which shall be called the *complete group of whirl-motions*, and denoted by Γ_6.

Let $\mathfrak{M}$ denote the magnification $(z, \zeta)\rightarrow(z^*, \zeta^*)$ given by

$$z^* = kz, \qquad \zeta^* = \zeta, \tag{5.5}$$

where k is real and positive. Let $\Gamma_7\equiv\mathfrak{M}\Gamma_6$. Evidently Γ_7 is a mixed seven-parameter group of turbine-preserving transformations comprising the eight continuous families: $\mathfrak{G}_7^i\equiv\mathfrak{M}\,\mathfrak{G}_6^i$, $\mathfrak{H}_7^i\equiv\mathfrak{M}\,\mathfrak{H}_6^i$, $i=1, 2, 3, 4$; Γ_7 shall be called the *complete group of whirl-similitudes*.

6. The kinematic representation. Owing to the (1, 1) correspondence subsisting between turbines $\mathfrak{T}$: $[l, r]$ and ordered pairs of complex numbers l, r, $\mathfrak{T}$ can be mapped upon the ordered pair of points on the Gauss plane represented by l and r in this order. This mapping, $\mathfrak{T}\rightarrow(l, r)$, of the ∞^4 turbines upon the ∞^4 ordered point pairs shall be called a *kinematic representation*. With the aid of equations (3.3) and (3.6), which entail necessary and sufficient conditions for the incidence of turbine and lineal element in the former case, and for the incidence of turbine and flat field in the latter, the following two theorems can now be established.

THEOREM 1. *The kinematic* (1, 1) *representation of turbines by ordered pairs of real points on a plane, namely* $\mathfrak{T}\leftrightarrow(l, r)$, *induces a* (1, 1) *correspondence between lineal elements* (z, ζ) *and planar euclidean displace-*

ments $l\to r$. To the elements (z, ζ) such that $\zeta=-1$ correspond the translations $r=l-z$, and to those such that $\zeta\neq -1$ correspond the rotations around the point $z/(\zeta+1)$ through the angle arg $(-\bar{\zeta})$.

THEOREM 2. *The kinematic* (1, 1) *representation* $\mathfrak{T}\leftrightarrow(l, r)$ *induces a* (1, 1) *correspondence between flat fields* $\{s, \sigma\}$ *and planar euclidean symmetries $l\to r$. To the flat fields $\{s, \sigma\}$ such that $\sigma\bar{s}+s=0$ correspond reflections $l\to r$ in the line $\bar{s}z+s\bar{z}=s\bar{s}$; and to those such that $\sigma\bar{s}+s\neq 0$ correspond symmetries compounded, in either order, of a reflection in the axis $z-s/2=\sigma(\bar{z}-\bar{s}/2)$ and a translation (parallel to the axis) given by $r=l-(\sigma\bar{s}+s)/2$.*

The proper whirl-similitudes—that is, those constituting the four families $\mathfrak{G}_7^i$—regarded as turbine transformations $\mathfrak{T}\to\mathfrak{T}^*$, are mapped by means of the kinematic representation upon pairs of similitudes $l\to l^*$, $r\to r^*$ having the same ratio of magnification. These pairs of similitudes form four continuous families corresponding to the $\mathfrak{G}_7^i$; their equations are as follows:

$$
\begin{aligned}
&\overset{1}{\mathfrak{G}}_7\colon\quad l^* = \alpha k l + a, && r^* = \beta k r + b,\\
&\overset{2}{\mathfrak{G}}_7\colon\quad l^* = \alpha k \bar{l} + a, && r^* = \beta k \bar{r} + b,\\
&\overset{3}{\mathfrak{G}}_7\colon\quad l^* = \alpha k l + a, && r^* = \beta k \bar{r} + b,\\
&\overset{4}{\mathfrak{G}}_7\colon\quad l^* = \alpha k \bar{l} + a, && r^* = \beta k r + b,\\
&\qquad k \text{ real and positive}, && |\alpha| = |\beta| = 1.
\end{aligned}
$$

By interchanging l and r in the right-hand members of the equations of the similitudes corresponding to $\mathfrak{G}_7^i$, we obtain the equations of the kinematic image of the family of improper whirl-similitudes $\mathfrak{H}_7^i$. The kinematic images of the eight families comprising Γ_6 are the pairs of euclidean motions to which the above pairs of similitudes reduce when $k=1$.

7. **The quasi-elliptic representation.** Let a point $\mathfrak{p}$ (X, Y, Z) in S_3 have the homogeneous coordinates $X=p_2/p_1$, $Y=p_3/p_1$, $Z=p_0/p_1$. Let a line C, together with the pair of conjugate imaginary points on C, namely:

$$
\begin{aligned}
c_l, &\quad p_0\!:\!p_1\!:\!p_2\!:\!p_3 = 0\!:\!0\!:\!i\!:\!1,\\
c_r, &\quad p_0\!:\!p_1\!:\!p_2\!:\!p_3 = 0\!:\!0\!:\!1\!:\!i,
\end{aligned}
$$

and the pair of conjugate imaginary planes through C:

$$
\gamma_l, \quad p_0 - ip_1 = 0, \qquad \gamma_r, \quad p_0 + ip_1 = 0
$$

be the Cayley absolute underlying a non-euclidean space Q_3, called *quasi-elliptic* by Blaschke [1, 2]. The proper (real) points of Q_3 are those that do not lie on C; the proper lines are those that do not intersect C; and the proper planes are those that do not pass through C.

By means of a method independently discovered by J. Grünwald [5] and W. Blaschke [1, 2], the ∞^4 proper (real) lines $\mathfrak{g}$ of Q_3 can be mapped continuously and (1, 1) upon the ∞^4 pairs of ordered real points (l, r) of the euclidean ground plane $Z=0$ in a manner such that the following two properties subsist: (1) the two fields of points l and r formed by the ∞^2 pairs (l, r) that correspond to the ∞^2 proper real lines $\mathfrak{g}$ passing through a proper real point $\mathfrak{p}$ are such that a unique euclidean displacement $l \rightarrow r$ will transform the points l into their associated points r; (2) the two fields l and r formed by the ∞^2 pairs (l, r) that correspond to the ∞^2 proper real lines $\mathfrak{g}$ lying in a proper real plane π are such that a unique euclidean symmetry $l \rightarrow r$ will transform the points l into their associated points r. By virtue of these two properties a (1, 1) correspondence exists between the points $\mathfrak{p}$ of Q_3 and the euclidean displacements in the ground plane on the one hand, and between the planes π and the euclidean symmetries in the ground plane on the other hand.

Let us associate with every proper real line $\mathfrak{g}$ in Q_3 a turbine $\mathfrak{T}$ in such a manner that $\mathfrak{g}$ is the B-G (Blaschke-Grünwald) image of that pair of points l, r as that of which $\mathfrak{T}$ is the kinematic image. The ensuing correspondence $\mathfrak{g} \leftrightarrow \mathfrak{T}$ is (1, 1) and continuous. Furthermore, let us establish a (1, 1) correspondence between lineal elements $\mathcal{E}$ and proper points $\mathfrak{p}$ in Q_3 such that $\mathcal{E}$ and $\mathfrak{p}$ are images—kinematic and B-G, respectively,—of the same euclidean displacement $l \rightarrow r$; and, finally, let us establish a third (1, 1) correspondence between flat fields $\mathcal{J}$ and proper planes π such that $\mathcal{J}$ and π are images of the same symmetry $l \rightarrow r$. These three (1, 1) correspondences define the *quasi-elliptic representation* of whirl-similitude geometry. Incidences among lineal elements, turbines, and flat fields imply corresponding incidences among their quasi-elliptic images.

Inasmuch as the group $\overline{\mathfrak{G}}_7^1$ of collineations in Q_3 that leave the points c_l, c_r and the planes γ_l, γ_r of the absolute individually invariant corresponds, by virtue of the B-G mapping, to the group of transformations $(l, r) \rightarrow (l^*, r^*)$ such that $l \rightarrow l^*$, $r \rightarrow r^*$ are two similitudes having the same ratio of magnification [1, 2], we see that, by virtue of the results given in §6, $\overline{\mathfrak{G}}_7^1$ is simply isomorphic to $\mathfrak{G}_7^1$. The totality of automorphisms (collineations and correlations) of Q_3 constitutes a mixed group $\overline{\Gamma}_7$, composed of eight continuous families, $\overline{\mathfrak{G}}_7^i$ and $\overline{\mathfrak{H}}_7^i$, $i=1, 2, 3, 4$, which are isomorphic to the families comprising the complete group of whirl-similitudes Γ_7. To characterize the isomorphism

$\Gamma_7 \leftrightarrow \overline{\Gamma}_7$, it is sufficient to indicate the quasi-elliptic representations of $\mathfrak{J}_1$, $\mathfrak{J}_2$, and $\mathfrak{J}_3$. These are, in terms of the involutory transformations effected on the absolute, as follows

$$\mathfrak{J}_1\colon \quad c_l \to c_r, \qquad c_r \to c_l, \qquad \gamma_l \to \gamma_r, \qquad \gamma_r \to \gamma_l;$$

$$\mathfrak{J}_2\colon \quad c_l \to \gamma_l, \qquad c_r \to \gamma_r, \qquad \gamma_l \to c_l, \qquad \gamma_r \to c_r;$$

$$\mathfrak{J}_3\colon \quad c_l \to c_l, \qquad c_r \to c_r, \qquad \gamma_l \to \gamma_r, \qquad \gamma_r \to \gamma_l.$$

To $\mathfrak{G}_6^1$ (Kasner's whirl-motion group) there corresponds in quasi-elliptic space, by virtue of the quasi-elliptic representation, the group of *quasi-motions* $\overline{\mathfrak{G}}_6^1$. The quasi-motions can be uniquely resolved into products of what Blaschke calls, because of their analogy to Clifford's left and right translations in elliptic space, *quasi-left and quasi-right* translations in Q_3 [1]; the former are the quasi-elliptic images of the displacements $\mathfrak{D}_3^+$, and the latter are the images of the direct whirls $\mathfrak{W}_3^+$.

By means of the quasi-elliptic representation, many of the results obtained by Kasner and DeCicco for the geometry of whirl-motions can be easily identified with results obtained independently by Blaschke and Grünwald for quasi-elliptic space.

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