

A NOTE ON THE LOWER SEMI-CONTINUITY OF DOUBLE INTEGRALS IN THE PARAMETRIC FORM

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Consider a double integral problem of the calculus of variations in parametric form, that is, a problem involving an integral of the form

$$\int \int_B f(x, X) du$$

where x stands for the three coordinate functions, $x^1(u^1, u^2)$, $x^2(u^1, u^2)$, $x^3(u^1, u^2)$, and X stands for their three Jacobians.

Various theorems of the following general type have been proved by McShane,¹ Caccioppoli,² Cimmino,³ and Radó:⁴

If (i) the triples of functions $x_n^i(u^1, u^2)$, $i=1, 2$; $n=0, 1, 2, \dots$, satisfy certain conditions; (ii) the surfaces defined by $x^i = x_n^i(u^1, u^2)$, $i=1, 2$; $n=1, 2, \dots$, converge in some prescribed sense to the surface defined by $x_0^i = x_0^i(u^1, u^2)$, $i=1, 2$; (iii) $f(x, X)$ satisfies certain conditions; then

$$\liminf_n \int \int_{B_n} f(x_n(u), X_n(u)) du \geq \int \int_{B_0} f(x_0(u), X_0(u)) du.$$

The conditions on $f(x, X)$ usually contain a condition on its Weierstrass E -function

$$E(x, X, \bar{X}) = f(x, \bar{X}) - \bar{X}^\alpha f_\alpha(x, X),$$

the summation convention being used for convenience in writing.

McShane¹ requires either $f > 0$, $E \geq 0$; or $f \geq 0$, $E > 0$. Caccioppoli² makes the same requirements. Cimmino³ requires only $f \geq 0$, $E \geq 0$, but works with smooth surfaces.

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¹ McShane, *On the semi-continuity of double integrals in the calculus of variations*, Annals of Mathematics, (2), vol. 33 (1932), pp. 460-484.

² Caccioppoli, *Gli integrali doppi di forma parametrica nel calcolo delle variazioni*, Arri del Reale Istituto Veneto di Scienze, Lettere ed Arti, Anno Accademico, 1933 1934, vol. 93, Part 2.

³ Cimmino, *Sulle condizioni necessarie e sufficienti per la semi-continuità degli integrali doppi di forma parametrica*, Annali di Mathematica Pura ed Applicata, vol. 15 (1936), pp. 159-173. The author wishes to thank Professor McShane for calling his attention to this paper.

⁴ Radó, *On the semi-continuity of double integrals in the parametric form*, Transactions of this Society, vol. 51 (1942), pp. 336-361. The author wishes to thank Professor Radó for access to the manuscript of this paper before it was published.

Radó⁴ proves the semi-continuity theorems under the same restrictions on $f(x, X)$ as McShane and Caccioppoli, but for a more general class of surfaces; namely, the class of surfaces that admit of a parametric representation for which the Lebesgue area is given by the usual integral formula.

The question arises as to whether we cannot prove the semi-continuity theorem for the class of surfaces which Radó considers with the requirements that $f \geq 0, E \geq 0$. We shall first prove a lemma on the Weierstrass E -function, then prove a stronger form of Lemma 1.5 in Radó.⁴ The remainder of the proof as given by Radó will then prove the semi-continuity theorem with $f \geq 0, E \geq 0$ if the revised form of Lemma 1.5 is used.

We shall use the notations and definitions in Radó⁴ throughout the remainder of the paper.

LEMMA 1. *If f is admissible and x_0, X_0 are such that*

- (1) $f(x_0, X_0) > 0,$
- (2) $f(x_0, X) \geq 0,$
- (3) $E(x_0, X_0, X) \geq 0,$

for every X , then there exists a $\delta > 0$ such that if $\bar{X}^\alpha f_\alpha(x_0, X_0) > 0$ then $f(x_0, \bar{X}) > \delta \|\bar{X}\|$.

PROOF. Since

$$X_0^\alpha f_\alpha(x_0, X_0) = f(x_0, X_0) > 0$$

at least one of $f_1(x_0, X_0), f_2(x_0, X_0), f_3(x_0, X_0)$ is different from 0. Hence the set of points \bar{X} satisfying $\bar{X}^\alpha f_\alpha(x_0, X_0) > 0, \|\bar{X}\| = 1$, is an open hemisphere H^0 . Since $E(x_0, X_0, \bar{X}) \geq 0$ we have $f(x_0, \bar{X}) > 0$ for every $\bar{X} \in H^0$. If we deny the lemma, then there exists a sequence $\{\bar{X}_n\}, \bar{X}_n \in H^0$ such that $f(x_0, \bar{X}_n) \rightarrow 0$. Hence there exists a point $\bar{X}_0 \in H - H^0$ such that $f(x_0, \bar{X}_0) = 0$.

Let $H - H^0 = C$, and let the plane containing C be P . Let $d(\bar{X}, P)$ denote the distance of \bar{X} from P . Then if $\bar{X} \in H$ we have

$$(*) \quad \bar{X}^\alpha f_\alpha(x_0, X_0) = d(\bar{X}, P) \left(\sum_{i=1}^3 f_i^2(x_0, X_0) \right)^{1/2}.$$

Let C' be the great circle with center at $X = 0$, unit radius, passing through \bar{X}_0 , and perpendicular to C . Let the direction cosines of the tangent to C' at \bar{X}_0 toward H^0 be $\cos \phi_1, \cos \phi_2, \cos \phi_3$.

Let $\partial f(x_0, \bar{X}_0) / \partial \nu$ be the directional derivative of f along this tangent. Then we have

$$\frac{\partial f(x_0, \bar{X}_0)}{\partial \nu} = f_\alpha(x_0, \bar{X}_0) \cos \phi_\alpha.$$

On the one hand, we have from (*)

$$\frac{\partial f(x_0, \bar{X}_0)}{\partial \nu} \geq \left(\sum_{i=1}^3 f_i^2(x_0, X_0) \right)^{1/2} > 0,$$

and on the other hand, since $f(x_0, \bar{X}_0) = 0, f(x_0, X) \geq 0$ we have

$$\frac{\partial f(x_0, \bar{X}_0)}{\partial \nu} = 0.$$

This is a contradiction. Hence the lemma is true.

LEMMA 2. *Let there be given an admissible f and a set of six constants $(x_0^1, x_0^2, x_0^3, X_0^1, X_0^2, X_0^3) = (x_0, X_0)$ such that*

- (a) $f(x_0, X_0) > 0,$
- (b) $f(x_0, X) \geq 0,$
- (c) $E(x_0, X_0, X) \geq 0,$

for all X . Then there exist two positive constants δ_1, δ_2 such that the following holds. If $\eta, \bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{X}^1, \bar{X}^2, \bar{X}^3$ satisfy the conditions

- (α) $0 < \eta < \delta_1,$
- (β) $\bar{X}^\alpha f_\alpha(x_0, X_0) > 0,$
- (γ) $|f(\bar{x}, \bar{X}) - f(x_0, \bar{X})| \leq \eta \|\bar{X}\|,$

then

$$f(\bar{x}, \bar{X}) - f(x_0, \bar{X}) + E(x_0, X_0, \bar{X}) \geq -\eta \delta_2 \bar{X}^\alpha f_\alpha(x_0, X_0).$$

PROOF. The conditions of Lemma 1 are satisfied. Hence there exists a positive constant δ as described there. Let $\delta_1 = \delta/2, \delta_2 = 2/\delta$.

Case 1. $\bar{X}^\alpha f_\alpha(x_0, X_0) \geq (\delta/2) \|\bar{X}\|$. Then

$$\begin{aligned} f(\bar{x}, \bar{X}) - f(x_0, \bar{X}) + E(x_0, X_0, \bar{X}) &\geq -\eta \|\bar{X}\| \\ &\geq -\eta \delta_2 \bar{X}^\alpha f_\alpha(x_0, X_0). \end{aligned}$$

Case 2. $\bar{X}^\alpha f_\alpha(x_0, X_0) < (\delta/2) \|\bar{X}\|$. Then

$$\begin{aligned} f(\bar{x}, \bar{X}) - f(x_0, \bar{X}) + E(x_0, X_0, \bar{X}) &= f(\bar{x}, \bar{X}) - \bar{X}^\alpha f_\alpha(x_0, X_0) \\ &\geq f(x_0, \bar{X}) - \eta \|\bar{X}\| - \bar{X}^\alpha f_\alpha(x_0, X_0) \\ &\geq 0 \\ &\geq -\eta \delta_2 \bar{X}^\alpha f_\alpha(x_0, X_0). \end{aligned}$$

Therefore the lemma is true.

Let K_2 be the class of oriented surfaces whose Lebesgue area is given by the usual integral formula. Let ${}_oS_0, {}_oS_n$ be oriented surfaces in the class K_2 such that:

- (I) ${}_o S_n \rightarrow {}_o S_0$.
- (II) If $T_0: x_0^i = x_0^i(u^1, u^2), (u^1, u^2) \in B_0, i = 1, 2, 3$ is a representation of ${}_o S_0$ then there exists, in x -space, a closed bounded set A such that $\sum (T_0, B_0) \subset A^0$, and $f(x, X) \geq 0$ for $x \in A$ and for every vector X .
- (III) For a. e. point $u \in B_0^0$ such that $X_0(u)$ exists and is different from 0 we have $E(x_0(u), X_0(u), \bar{X}) \geq 0$ for every vector $\bar{X} \neq 0$.

As we mentioned above the proof given by Radó will now lead to the following result.

THEOREM. *If f is admissible and if conditions (I), (II), and (III) are satisfied then*

$$\liminf_n \int \int_{B_n} f(x_n(u), X_n(u)) du \geq \int \int_{B_0} f(x_0(u), X_0(u)) du.$$

This theorem is final in a definite sense. We cannot enlarge the class of surfaces if we use ordinary Jacobians. Counterexamples have been given which show the necessity of the conditions $E \geq 0, f \geq 0$. Moreover if f is only assumed to be continuous, convex, and non-negative, then we do not necessarily have lower semi-continuity. For, let

$$f(x, X) = |X^3 + g(x^1, x^3)X^1|$$

where

$$g(x^1, x^3) = \frac{\cos 16x^1}{2} \quad \text{if } x^3 > 5/8,$$

$$g(x^1, x^3) = \frac{\cos n^4 x^1}{n} \quad \text{if } 1/n - 1/n^3 \leq x^3 \leq 1/n + 1/n^3, n \geq 2,$$

$$g(x^1, x^3) = \frac{\cos (n + 1)^4 x^1}{n + 1} + \frac{x^3 - (1/(n + 1) + 1/(n + 1)^3)}{(1/n - 1/n^3) - (1/(n + 1) + 1/(n + 1)^3)} \cdot \left[\frac{\cos n^4 x^1}{n} - \frac{\cos (n + 1)^4 x^1}{n + 1} \right]$$

$$g(x^1, x^3) = 0 \quad \text{if } 1/(n + 1) + 1/(n + 1)^3 < x^3 < 1/n - 1/n^3, \text{ if } x^3 \leq 0.$$

Then f satisfies all the conditions stated. Further let

$$x_0^1 = u^1, \quad x_n^1 = u^1, \quad 0 \leq u^1 \leq \pi,$$

$$\begin{aligned} x_0^2 &= u^2, & x_n^2 &= u^2, & 0 &\leq u^2 \leq 1, \\ x_0^3 &= 0, & x_n^3 &= \frac{1}{n} + \frac{\sin n^4 u^1}{n^3}. \end{aligned}$$

Then we have

$$\begin{aligned} \liminf_n \iint_{B_n} f(x_n, X_n) du &= \liminf_n \int_0^\pi \int_0^1 |1 - \cos^2 n^4 u^1| du^2 du^1 \\ &= \frac{\pi}{2} < \pi = \int_0^\pi \int_0^1 du^2 du^1 \\ &= \iint_{B_0} f(x_0, X_0) du. \end{aligned}$$

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A NON-INVOLUTORIAL SPACE TRANSFORMATION ASSOCIATED WITH A $Q_{1,n}$ CONGRUENCE

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1. **Introduction.** The involutorial transformation associated with the congruence of lines meeting a curve of order m and an $(m-1)$ -fold secant has been studied by DePaolis,¹ and Vogt² has studied the non-involutorial transformations for a linear congruence and bundle of lines. Cunningham³ has recently studied some non-involutorial transformations associated with a $Q_{1,2}$ congruence. In the present paper a non-involutorial transformation associated with the congruence of lines on a plane curve of order n having an $(n-1)$ -point and a secant through that point is considered. The bundle of lines through the multiple point is not considered as belonging to the congruence. The tangents to the curve at the point are considered to be distinct.

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¹ DePaolis, *Alcuni particolari trasformazioni involutori dello spazio*, Rendiconti dell'Accademia dei Lincei, Rome, (4), vol. 1 (1885), pp. 735-742, 745-758.

² Vogt, *Zentrale und windschiefe Raum-Verwandtschaften*, Jahresbericht der Schlesischen Gesellschaft für Vaterlandische Kultur, class 84, 1906, pp. 8-16.

³ Cunningham, *Non-involutorial space transformations associated with a $Q_{1,2}$ congruence*, this Bulletin, vol. 47 (1941), pp. 309-312.