ON THE MAPPING OF n QUADRATIC FORMS

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If $Q_1(z), Q_2(z), \cdots, Q_n(z)$ are quadratic forms in the real variables $z^1, z^2, \cdots, z^m$ with real coefficients, the question arises as to the conditions under which there exists a linear combination

$$\lambda_1 Q_1(z) + \lambda_2 Q_2(z) + \cdots + \lambda_n Q_n(z)$$

which is positive definite. A number of recent papers have considered this question.

For the case $n=2$ a satisfactory answer was obtained by Paul Finsler,[1] and independently by a group of interested persons at the University of Chicago.[2]

For any finite n, Finsler obtained sufficient conditions under the restriction that the number of independent variables does not exceed four. The conditions are quite involved, and are in terms of a certain type of algebraic manifold which Finsler designates "Freigebilde" and of which he had made an elaborate study in an earlier paper.

For any finite n and any finite number of independent variables Hestenes and McShane, in a joint paper,[3] obtained sufficient conditions. They are obviously not necessary, and though they seem exactly suited to the application which the authors desired to make, their lack of symmetry perhaps leaves something to be desired.

In a recent paper[4] the present author called attention to the suitability of the theory of convexity as a means for studying this type of question, and treated the case of two quadratic forms in m variables from this point of view. The purpose of the present paper is to make a similar study of the general case of n quadratic forms in m variables.

1. **The map $\mathfrak{M}$.** The n quadratic forms $Q_1(z), Q_2(z), \cdots, Q_n(z)$ may be thought of as mapping the m-dimensional space $\mathfrak{Z}_m$ of points $z=(z^1, z^2, \cdots, z^m)$ onto an n-dimensional space $\mathfrak{X}_n$ of points

Presented to the Society, December 31, 1941; received by the editors September 5, 1941.

[1] *Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratischer Formen*, Commentarii Mathematici Helvetici, vol. 9 (1937), pp. 188–192.

[2] From this group two different proofs of the essential theorem appeared: one by A. A. Albert, this Bulletin, vol. 44 (1938), pp. 250–253; and one by W. T. Reid, ibid., pp. 437–440.

[3] *A theorem on quadratic forms and its application in the calculus of variations*, Transactions of this Society, vol. 47 (1940), pp. 501–512.

[4] *On the mapping of quadratic forms*, this Bulletin, vol. 47 (1941), pp. 494–498.

$x = (x^1, x^2, \cdots, x^n)$. The map $\mathfrak{M}$ consists of the set of points x defined by

$$x = (Q_1(z), Q_2(z), \cdots, Q_n(z)), \qquad z \text{ on } \mathfrak{Z}_m,$$

a formula which may be condensed to

$$\mathfrak{M}: \quad x = Q(z), \qquad z \text{ on } \mathfrak{Z}_m.$$

In view of the homogeneity property

$$(1) \qquad Q(tz) = t^2 Q(z)$$

the map consists of half-lines emanating from the origin $x = (0)$. In the particular case $n = 2$ it is convex, as was shown in the earlier paper. For $n > 2$ it is not necessarily convex. For example, the map of three quadratic forms in two variables lies on a conical surface. It is not necessarily closed, but it may be shown to be closed if the forms $Q_i(z)$ have no common zero except $z = (0)$. However, more useful for our present purpose than the map $\mathfrak{M}$ itself is a certain subset of it which we now define.

2. **The map $\mathfrak{M}_1$.** By $\mathfrak{M}_1$ we shall denote the subset consisting of those points of $\mathfrak{M}$ which correspond to points z on the unit $(m-1)$-sphere in $\mathfrak{Z}_m$. The formula for the points of $\mathfrak{M}_1$ is then

$$\mathfrak{M}_1: \quad x = Q(z), \qquad \|z\| = 1.$$

The particular usefulness of the map $\mathfrak{M}_1$ is due to the facts that it is *bounded* and *closed*, and that it contains the origin $x = (0)$ only in exceptional cases. That it is bounded follows immediately from the continuity of $Q(z)$ on the $(m-1)$-sphere. That it is closed may be seen as follows.

Suppose $\{x_j\}$ is a sequence of points in $\mathfrak{M}_1$ defined by

$$(2) \qquad x_j = Q(z_j), \qquad \|z_j\| = 1,$$

and suppose there is in the space $\mathfrak{X}_n$ a point $\bar{x}$ such that

$$(3) \qquad \bar{x} = \lim_{j \to \infty} x_j.$$

We will prove that $\bar{x}$ belongs to $\mathfrak{M}_1$.

The sequence of points $\{z_j\}$ on the $(m-1)$-sphere $\|z\| = 1$ has an accumulation point $\bar{z}$, which must itself be on the $(m-1)$-sphere (that is, $\|\bar{z}\| = 1$), since otherwise it would be a definite distance from it. And from the sequence $\{z_j\}$ there can be chosen a subsequence $\{z_{j_k}\}$ such that

$$\lim_{k\to\infty} z_{j_k} = \bar{z}. \tag{4}$$

Then from (2), (3), (4), and the continuity of $Q(z)$ it follows that $\bar{x}=Q(\bar{z})$, and hence $\bar{x}$ belongs to $\mathfrak{M}_1$.

3. **The convex extension of $\mathfrak{M}_1$.** The convex extension of a set of points may be defined as the smallest convex set which contains it, or (equivalently) as consisting of those points which can be the centroids of positive mass distribution (of total mass unity) at suitably chosen points of the given set. According to the latter definition, $C(\mathfrak{M}_1)$, the convex extension of $\mathfrak{M}_1$, consists of the points represented by

$$C(\mathfrak{M}_1):\quad x = \sum_{j=1}^{r} m_j Q(z_j)$$

where

$$\|z_j\| = 1,\ m_j > 0,\quad \textit{and}\quad \sum_{j=1}^{r} m_j = 1. \tag{5}$$

It may also be remarked that whenever a representation (5) exists, there exists an equivalent one in which the number r of terms in the summation does not exceed $n+1$.

4. **On the existence of a definite $\sum \lambda_i Q_i(z)$.** It is well known that through each point exterior to the convex extension of a bounded and closed set of points there passes a bounding plane[5] (Schranke) for the set, while through no point of the convex extension does there pass such a plane. Hence if the origin $x=(0)$ does not belong to $C(\mathfrak{M}_1)$, there is a plane $\sum_{i=1}^{n}\lambda_i x^i=0$ such that

$$\sum_{i=1}^{n} \lambda_i Q_i(z) > 0 \quad \textit{whenever}\quad \|z\| = 1, \tag{6}$$

while if $x=(0)$ does belong to $C(\mathfrak{M}_1)$ there is no such plane. And in view of the homogeneity relation (1), the validity of (6) is obviously equivalent to the validity of the same relation with z restricted only by $z\neq(0)$.

The condition that $x=(0)$ shall belong to $C(\mathfrak{M}_1)$ is that there shall exist valid relations

$$\sum_{j=1}^{r} m_j Q_i(z_j) = 0, \qquad i = 1, 2, \cdots, n, \tag{7}$$

[5] For simplicity we use the term "plane" to designate the locus of a linear equation in the x-coordinates, regardless of the dimensions of the space $\mathfrak{X}_n$.

where the z_j and m_j are restricted as in (5). And again in view of the homogeneity property (1) and the homogeneity of (7) in the m_j, the only essential restrictions are $z_j \neq (0)$ and $m_j > 0$. We have then the following theorem:

THEOREM 1. *A necessary and sufficient condition that there exist a linear combination* $\sum \lambda_i Q_i(z)$ *such that*

$$\sum_{i=1}^{n} \lambda_i Q_i(z) > 0 \quad \textit{when} \quad z \neq (0)$$

is that there exist in $\mathfrak{Z}_m$ *no set of points* $z_j \neq (0)$, $(j = 1, 2, \cdots, r)$ *such that*

$$\sum_{j=1}^{r} m_j Q_i(z_j) = 0, \qquad i = 1, 2, \cdots, n, \tag{8}$$

the coefficients m_j *being positive.*

5. **On the existence of a semi-definite** $\sum \lambda_i Q_i(z)$. We have seen that when $x = (0)$ is exterior to $C(\mathfrak{M}_1)$ there exist definite linear combinations $\sum \lambda_i Q_i(z)$. It is easily seen that under these circumstances there also exist *semi*-definite linear combinations. We seek here however the more critical conditions under which there exists a linear combination which is semi-definite but none which is definite. This is the case when $x = (0)$ is a *boundary* point of $C(\mathfrak{M}_1)$. For in that case there passes through $x = (0)$ a *supporting* plane (Stützebene) $\sum \lambda_i x^i = 0$ for $\mathfrak{M}_1$, such that

$$\sum_{i=1}^{n} \lambda_i Q_i(z) \geqq 0, \qquad z \text{ on } \mathfrak{Z}_m, \tag{9}$$

the restriction $\|z\| = 1$ being removable in view of the homogeneity property (1). The "=" sign in (9) is required for at least one point $z \neq (0)$, and hence the linear combination is truly semi-definite unless the $Q_i(z)$ are linearly dependent. We may now state this theorem:

THEOREM 2. *If the quadratic forms* $Q_i(z)$ *are linearly independent, necessary and sufficient conditions that they admit a linear combination which is semi-definite but none which is definite are*: (a) *that there exist in* $\mathfrak{Z}_m$ *a set of points* $z_j \neq (0)$, $(j = 1, 2, \cdots, r)$ *and associated positive constants* m_j *such that*

$$\sum_{j=1}^{r} m_j Q(z_j) = 0,$$

and (b) *that there exist a set of real constants* $a=(a^1, a^2, \cdots, a^n)$ *not representable in the form*

$$a = \sum_{j=1}^{r} m_j' Q(z_j'), \tag{10}$$

each m_j' *being positive and each* $z_j' \neq (0)$.

We have already seen that condition (a) is the condition that $x=(0)$ belong to $C(\mathfrak{M}_1)$. Condition (b) is necessary and sufficient to assure that it be a *boundary* point.

It is sufficient, for if some point a of $\mathfrak{X}_n$ is not representable in form (10), then no point on the half-line $x=at$, $(t>0)$, can belong to $C(\mathfrak{M}_1)$, and the origin $x=(0)$ can be approached by a sequence of exterior points on this half-line.

It is necessary, for in its absence every point a of the space $\mathfrak{X}_n$ is representable in form (10), and as a result some point on every half-line $x=at$, $(t>0)$, belongs to $C(\mathfrak{M}_1)$. And since the origin $x=(0)$ itself belongs to $C(\mathfrak{M}_1)$, it follows that there is a hypercube with this origin at its center and all of its vertices belonging to $C(\mathfrak{M}_1)$. The origin would then be an inner point of $C(\mathfrak{M}_1)$.

6. **Generalization.** Of the various properties of quadratic forms, only their continuity and their quadratic homogeneity were used in the above discussion. Indeed it remains valid if: (1) the m-dimensional space $\mathfrak{Z}_m$ is replaced by any normed linear space $\mathfrak{Z}$ in which the points on the unit hypersphere $\|z\|=1$ form a compact subspace; and (2) the n functions $Q_i(z)$ are real-valued functions on $\mathfrak{Z}$, continuous on the hypersphere $\|z\|=1$, and having the property that for real t, $Q_i(tz) = p(t)Q_i(z)$, where $p(t)$ is a non-negative real-valued function which vanishes when and only when $t=0$ and which has a real-valued inverse $t(p)$, not necessarily unique, but defined for all positive p.

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