ON 3-DIMENSIONAL MANIFOLDS

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Let P be a 3-dimensional manifold.[1] Let Q be a 2-dimensional manifold imbedded in P. Moreover, let P and Q admit of a *permissible simplicial division* K, that is, a simplicial division of P such that some subcomplex of K, say L, is a simplicial division of Q. Let K_i and L_i denote the ith normal subdivisions of K and L, respectively. We define the *neighborhood* N_i of L_i to be the simplicial complex consisting of the simplexes of K_i that have at least one vertex in L_i together with the sides of all such simplexes. By the *boundary* B_i of N_i we mean the simplicial complex consisting of the simplexes of N_i that have no vertex in L_i. Our purpose is to prove the following theorem.

THEOREM. *The boundary B_2 is a two-fold but not necessarily connected covering of Q, and change of permissible division K replaces B_2 by a homeomorph of itself.*

PROOF. The neighborhood N_1 is the sum of a set of 3-dimensional simplexes. Some of these 3-simplexes, say $a_1, a_2, \cdots$, have exactly one vertex in L_1, others, say $b_1, b_2, \cdots$, have exactly two vertices in L_1, while the remaining, say $c_1, c_2, \cdots$, have three vertices in L_1. Since K_1 is a normal subdivision of K, the intersection of L_1 and b_i or c_i is a 1-simplex or 2-simplex, respectively. Let α_i, β_i, and γ_i be the intersections of B_2 and a_i, b_i, and c_i, respectively. We shall regard α_i and γ_i as triangles with vertices on the 1-simplexes of a_i and c_i. Also we shall regard β_i as a square with vertices on the 1-simplexes of b_i.

Any 2-simplex of L_1, say ABC, is incident to exactly two of the c_i. Let $c_1 = ABCM$. There is a unique 3-simplex of N_1, say σ, that is incident to ABM and different from c_1. This σ is either a c_i, say c_2, or a b_i, say b_2. If σ is c_2, then the triangles γ_1 and γ_2 have a common side. Suppose that σ is $b_2 = ABMN$. The 2-simplex ABN is incident to a unique 3-simplex of N_1, say τ, with $\tau \neq ABMN$. This τ is either c_3 or b_3. If $\tau = b_3$, there is a c_4, or b_4. Finally we must find a $c_p = ABDS$, D in L_1, S in B_1. We now consider $\beta_2, \beta_3, \cdots$, and β_{p-1}. The sum of these squares is topologically equivalent to a square. One side of the square is coincident with a side of γ_1 and the opposite side coincident with a side of γ_p.

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[1] Our terminology is that of Seifert-Threlfall, *Lehrbuch der Topologie*. Manifolds are finite, while simplexes and cells are closed point sets.

Since K_1 is a manifold, we can repeat the construction and associate with ABC and ABD a second pair of triangles in B_2 that are either incident along a common side or incident to opposite sides of a square. But there is not a third such configuration associated with ABC and ABD. We repeat the construction for all pairs of adjacent 2-simplexes of L_1. Then to each 2-simplex of L_1 there correspond two triangles in B_2. Moreover, if two 2-simplexes of L_1 are incident along a side, the four corresponding triangles can be paired so that the two triangles of each pair either have a common side or are incident to opposite sides of a square.

Since P and Q are 3- and 2-manifolds, respectively, we can say that Q is two-sided in P in the neighborhood of any point of Q. Moreover, the two γ's of B_2 that correspond to a 2-simplex of L_1 lie on opposite sides of Q (in the neighborhood of this 2-simplex).

Consider a vertex X of L_1 and the 2-simplexes Δ_i of L_1 that have X as a vertex. On one side of Q (in the neighborhood of X) there corresponds to each Δ_i a unique γ_i, and the γ's have the same incidences as the corresponding Δ's (we say that two γ's are incident if they are incident to opposite sides of a square). Let us denote by R the points of these γ's and the squares incident to pairs of these γ's. Let A denote the points of all α_i's that are in a_i's incident to X and on the side of Q that we are considering.

We shall show that $R+A$ is a 2-cell. To do this we shall show that $R+A$ is a manifold relative to its boundary, that its boundary consists of one or more circles, and that any 1-cycle of $R+A$ bounds in $R+A$. First we observe that B_2 is a manifold; this fact follows from the structure of B_2 and the fact that K_1 is a manifold; the argument is elementary and we omit it. Since $R+A$ is the sum of 2-cells α, β, and γ, the set $R+A$ is a manifold relative to its boundary.

To show that this boundary of $R+A$ consists of one or more circles we shall study the incidences among the cells of $R+A$. First, let a_i have X as a vertex. If a 2-dimensional side of a_i is not in B_1, this side must be a side of an a_j or b_j. Furthermore, this a_j or b_j has X as a vertex. Hence, any side of an α_i is also a side of an α_j or β_j of $R+A$. Next, let c_i have vertices $XABM$, M in B_1. The sides of γ_i that are in XAM and XBM are sides of γ_j's or β_j's of $R+A$. But the side of γ_i in ABM is not incident to any other 2-cell of $R+A$. This side is part of the boundary of $R+A$. Finally, let b_i have vertices $XAMN$, A in L_1. The sides of β_i in XAM and XAN are incident to sides of β_j's or γ_j's of $R+A$; the side of β_i in XMN is incident to an α_j or β_j of $R+A$; but the side of β_i in AMN is not incident to any other 2-cell of $R+A$. This side is part of the boundary of $R+A$. Examination of the segments of the boundary of $R+A$ shows that they fit together to form one or more circles.

We next show that if C is a 1-dimensional cycle of $R+A$, then C bounds in $R+A$. We shall find it convenient to replace A by a new set that will never be empty. We define A' to be A together with all vertices of γ's of R that are not in the boundary of $R+A$ and all sides of squares of R that are not sides of γ's of R and not in the boundary of $R+A$. If A is not empty, the set A' is the same as A. But in any case A' is not empty, and $R+A'$ is the same set as $R+A$. The set $(R+A')-\overline{A}'$ is homeomorphic to a 2-cell with an inner point removed because $(R+A')-\overline{A}'$ can be obtained from the configuration of the 2-simplexes of L_1 that have X as a vertex by removing X and replacing some of the 1-simplexes by squares (open along one side). Hence, the cycle C is homologous in $R+A'$ to a cycle on A', and we assume that C is on A'. The set A' is part of b, the boundary of the combinatorial neighborhood of X in K_2. Since K_2 is a manifold, the set b is a 2-sphere. Assume that C does not bound in A'. Then C must surround a 2-simplex of b that is not in A'. We easily find a 2-simplex of $R+A'$ that is not incident along one of its sides to another 2-simplex of the manifold B_2. This contradiction proves that C bounds, and the proof that $R+A$ is a 2-cell is complete.

Now we draw some lines on $R+A$. If two γ's have a common side, we draw a line coincident with this common side. If two γ's are incident to a square, we draw a line across the square half way between the γ's. All these lines are continued so that they meet at a point of A. These lines give a subdivision of $R+A$ that is combinatorially equivalent to the combinatorial neighborhood of X in L_1. The lines can be drawn for all $R+A$ of B_2 and we get a subdivision of B_2 that is combinatorially equivalent to a two-fold but not necessarily connected covering of L_1.

A triangle of the covering is associated with a 2-simplex of L_1 and a side of Q (in the neighborhood of this simplex). Hence, a homeomorphism is determined between this covering and any covering obtained by changing the permissible division K.

The theorem is not true with B_1 rather than B_2. For example, let Q be the boundary of a 3-simplex of K. Then B_1 is a sphere and a point.

We can prove the following theorem in the same way but with much less effort.

THEOREM. *The above theorem is true if P and Q are replaced by* 2- *and* 1-*dimensional manifolds.*

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