finally an analytic r-cell contained in $\mathfrak{g} \cap W$. Hence $\mathfrak{g}$ contains a nucleus of G and hence $\mathfrak{g} = G$, a contradiction which proves the theorem.[3]

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[3] We have proved, incidentally, that if an everywhere dense subgroup $\mathfrak{g}$ of a simple Lie group G_r $(r>1)$ contains an analytic arc, then $\mathfrak{g} = G$.

VECTOR SPACES OVER RINGS

C. J. EVERETT[1]

1. **Introduction.** Let $\mathfrak{M} = u_1K + \cdots + u_mK$ be a vector space (linear form modul [5, p. 111]) over a ring $K = \{0, \alpha, \beta, \cdots ; \epsilon \text{ unit element}\}$. By a *submodul* $\mathfrak{N} \leqq \mathfrak{M}$ is meant an "admissible" submodul: $\mathfrak{N}K \leqq \mathfrak{N}$. Elements $v_1, \cdots, v_n$ of a submodul $\mathfrak{N}$ form a *basis* for $\mathfrak{N}$ (notation: $\mathfrak{N} = v_1K + \cdots + v_nK$) in case $\sum v_i\alpha_i = 0$ implies $\alpha_i = 0$, $i = 1, \cdots, n$, and if every element of $\mathfrak{N}$ is expressible in the form $\sum v_i\alpha_i$, $\alpha_i \in K$. The equivalent formulations of the ascending chain condition for submoduls of a vector space, and for right ideals of a ring will be used without further comment [5, §§80, 97].

2. **Basis number, linear transformations.** We remark that the following holds.

(A) *The ascending chain condition is satisfied by the submoduls of a vector space $\mathfrak{M}$ over K if and only if it is satisfied by the right ideals of K.*

An infinite chain of right ideals $\mathfrak{r}_1 < \mathfrak{r}_2 < \cdots$ in K yields an infinite chain of submoduls $u_1\mathfrak{r}_1 < u_1\mathfrak{r}_2 < \cdots$ in $\mathfrak{M}$. The other implication is proved in [5, p. 87].

[By using a lemma due to N. Jacobson (*Theory of Rings*, in publication) Theorem (A) and the corresponding theorem for descending chain condition are easily proved in a unified manner.]

Linear transformations of $\mathfrak{M}$ on $\mathfrak{M}$ are given by $u_j \rightarrow u_j' = \sum u_i\alpha_{ij}$. Write $(u_1', \cdots, u_m') = (u_1, \cdots, u_m)A$, $A = (\alpha_{ij})$. Under $u_j \rightarrow u_j'$, let $\mathfrak{M}_0 \rightarrow 0$. Thus $\mathfrak{M}/\mathfrak{M}_0 \cong \mathfrak{M}A \leqq \mathfrak{M}$. Clearly $\mathfrak{M}_0 = 0$ if and only if $Av = 0$ implies $v = 0$, v an $m \times 1$ matrix over K, and $\mathfrak{M}A = \mathfrak{M}$ if and only if there exists an $m \times m$ matrix R with $AR = I$, the identity matrix.

Possibilities (i) $\mathfrak{M}_0 = 0$ and $\mathfrak{M}A = \mathfrak{M}$; (ii) $\mathfrak{M}_0 > 0$ and $\mathfrak{M}A < \mathfrak{M}$; (iii) $\mathfrak{M}_0 = 0$ and $\mathfrak{M}A < \mathfrak{M}$ are familiar. The possibility of (iv) $\mathfrak{M}_0 > 0$

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and $\mathfrak{M}A = \mathfrak{M}$ is demonstrated later in (D), thus settling a question raised by van der Waerden [5, p. 115].

Case (iii) *implies an infinite descending chain in* $\mathfrak{M}$, *case* (iv) *an infinite ascending chain in* $\mathfrak{M}$.

(B) *The set* $(v_1, \cdots, v_n) = (u_1, \cdots, u_m)A$, $n < m$, *forms a basis for* $\mathfrak{M} = u_1K + \cdots + u_mK$ *if and only if the* $m \times m$ *matrix* $(A0)$ *has a right inverse*: $(A0)R = I$, *and* $Av = 0$ *implies* $v = 0$, v *an* $n \times 1$ *matrix over* K.

This is an immediate consequence of the basis definition.

(C) *If the right ideals of* K *satisfy the ascending chain condition, every basis of a vector space* $\mathfrak{M} = u_1K + \cdots + u_mK$ *has* m *elements.*

For a matrix $(A0)$ of the type in (B) defines a linear transformation of type (iv) violating the chain condition in K.

Hence with every vector space $\mathfrak{M}$ over a ring K with ascending chain condition for right ideals is associated a unique *basis number* $b(\mathfrak{M})$. K a quasi-field is a trivial special case.

(D) *If* K *is the ring of all infinite matrices over a field, with only a finite number of nonzero elements in each row and each column, then the vector space* $\mathfrak{M} = u_1K + \cdots + u_mK$, $m > 1$, *has a basis of one element*: $\mathfrak{M} = uK$. *Thus there exist, for arbitrary* m, $1 \times m$ *matrices* $(\alpha_1, \cdots, \alpha_m)$, $(\beta_1, \cdots, \beta_m)$ *over* K *such that* $(\alpha_1, \cdots, \alpha_m)'(\beta_1, \cdots, \beta_m) = I$, *the* $m \times m$ *identity matrix, with* $\alpha_i\beta = 0$, $i = 1, \cdots, m$, $\beta \in K$ *implying* $\beta = 0$.[2]

Let δ_i be the vector $(0, 0, \cdots, 0, 1, 0, \cdots)'$ with 1 in the ith position from above. Matric elements of K are defined by their column vectors; let the unit of K be $\epsilon = (\delta_1, \delta_2, \cdots)$ and $\alpha_1 = (0, \delta_1, 0, \delta_2, 0, \delta_3, \cdots)$, $\alpha_2 = (\delta_1, 0, \delta_2, 0, \delta_3, 0, \delta_4, \cdots)$, $\alpha_3 = \alpha_1'$, $\alpha_4 = \alpha_2'$. Let

$$A = \begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \alpha_3 & \alpha_4 \\ 0 & 0 \end{pmatrix}.$$

Then $AB = I$, and $\alpha_1\beta = \alpha_2\beta = 0$ implies $\beta = 0$, $\beta \in K$. Let

$$A_1 = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix},$$

where I is the $(m-2) \times (m-2)$ identity matrix. It follows from (B) that $u_1, \cdots, u_{m-2}, v$ form a basis for $\mathfrak{M}$, where $(u_1, \cdots, u_{m-2}, v, 0)$

[2] A' means A transpose.

$=(u_1, \cdots, u_m)A_1$. The induction is obvious, and $\mathfrak{M}$ has a basis of a single element. The theorem follows from (B).

3. **Vector spaces over right principal ideal rings.** We now remark that the following holds:

(E) *If* $\mathfrak{M}=u_1K+\cdots+u_mK$ *is a vector space over a ring* K *in which every right ideal* $\mathfrak{r}>0$ *is of type* ρ_0K, *where* $\rho_0\alpha=0$, $\alpha\in K$ *implies* $\alpha=0$, *then every submodul* $\mathfrak{N}$, $0<\mathfrak{N}\leqq\mathfrak{M}$, *has a basis of* n *elements*, $n\leqq m$.

This is only a trivial modification of the van der Waerden result [**5**, pp. 88, 121], appropriate since the condition subsequently also appears to be necessary (see (F)).

LEMMA 1. *If every submodul* $\mathfrak{N}$, $0<\mathfrak{N}\leqq\mathfrak{M}=u_1K+\cdots+u_mK$ *has a basis of* $n\leqq m$ *elements, and* $\mathfrak{r}$ *is a right ideal of* K, $0<\mathfrak{r}\leqq K$, *then the submodul* $\mathfrak{N}=u_1\mathfrak{r}\cup\cdots\cup u_m\mathfrak{r}$, *consisting of all sums* $\sum u_i\rho_i$, $\rho_i\in\mathfrak{r}$, *has a basis* $u_{11}, \cdots, u_{m1}$ *with* $u_1\mathfrak{r}=u_{i1}K$, $i=1, \cdots, m$, *and* u_{i1} *is a basis for* $u_i\mathfrak{r}$.

For $0<u_i\mathfrak{r}=u_{i1}K+\cdots+u_{in_i}K$, $1\leqq n_i\leqq m$, and $\mathfrak{N}=u_1\mathfrak{r}\cup\cdots\cup u_m\mathfrak{r}$ is a submodul for which the u_{ij} together form a basis of $\sum n_i$ elements. The hypothesis of the lemma implies the ascending chain condition in $\mathfrak{M}$, and hence in K (by (A)). Hence by (C) the basis number for $\mathfrak{N}$ is unique and $m\geqq\sum n_i\geqq m$, $n_i=1$, $i=1, \cdots, m$. Thus $u_i\mathfrak{r}=u_{i1}\cdot K$.

(F) *Let* $\mathfrak{M}=u_1K+\cdots+u_mK$ *be a vector space over* K. *Then every submodul* $\mathfrak{N}$, $0<\mathfrak{N}\leqq\mathfrak{M}$, *has a basis of* $n\leqq m$ *elements, if and only if every right ideal* $\mathfrak{r}>0$ *in* K *is of type* ρ_0K, *where* $\rho_0\alpha=0$, $\alpha\in K$, *implies* $\alpha=0$.

For if $\mathfrak{r}>0$ is a right ideal of K, by the lemma, $u_1\mathfrak{r}=u_{11}K$, $u_{11}=u_1\rho_0$, $\rho_0\in\mathfrak{r}$. Then $u_1\mathfrak{r}=u_1\rho_0K$ and $\mathfrak{r}=\rho_0K$. Moreover $\rho_0\alpha=0$ implies $u_{11}\alpha=0$ and $\alpha=0$.

Now suppose $\mathfrak{M}=u_1K+\cdots+u_mK$ is a vector space over a ring K of the type in (F). To every submodul $\mathfrak{N}$, $0<\mathfrak{N}\leqq\mathfrak{M}$, corresponds a unique basis number $b(\mathfrak{N})$. Define $b(0)=0$.

(G) *If* $\mathfrak{M}=u_1K+\cdots+u_mK$ *is a vector space over a ring* K *of the type in* (F), *the basis number* $b(\mathfrak{N})$, $0\leqq\mathfrak{N}\leqq\mathfrak{M}$, *is a positive modular functional* [**1**, p. 40]:

M1. $b(\mathfrak{A}\cup\mathfrak{B})+b(\mathfrak{A}\cap\mathfrak{B})=b(\mathfrak{A})+b(\mathfrak{B})$,

M2. $\mathfrak{A}\leqq\mathfrak{B}\leqq\mathfrak{M}$ *implies* $b(\mathfrak{A})\leqq b(\mathfrak{B})$.

M2 is clear from (F). A proof of M1 may be made by induction on $b(\mathfrak{A})$. We treat here only the following case:

Let K be a (noncommutative) domain of integrity in which every right ideal is principal.[3] The vector space $\mathfrak{M} = u_1K + \cdots + u_mK$ may then be regarded as imbedded in the vector space $\mathfrak{M}^* = u_1\overline{K} + \cdots + u_m\overline{K}$ where $\overline{K}$ is the quotient quasi-field of K. The existence of $\overline{K}$ follows from theorems developed by Ore [**3**, p. 466] and a proof by Teichmüller [**4**] that the least common multiple of nonzero elements in such a K is not zero. The correspondence

$$(\gamma)\ \mathfrak{N} = v_1K + \cdots + v_nK \rightarrow \mathfrak{N}^* = v_1\overline{K} + \cdots + v_n\overline{K}$$

is a well-defined correspondence on the lattice L of all K-submoduls of $\mathfrak{M}$ to the entire lattice $\overline{L}$ of $\overline{K}$-submoduls of $\mathfrak{M}^*$, (since $\mathfrak{N}^*$ is independent of the $\mathfrak{N}$-basis). Observe that $b(\mathfrak{N}) = b(\mathfrak{N}^*)$ as a submodul of $\mathfrak{M}^*$. For the K-independence of a basis $(v_1, \cdots, v_n)$ of $\mathfrak{N}$ implies the $\overline{K}$-independence of $v_1, \cdots, v_n$: Let $\sum v_i\bar{\alpha}_i = 0$, $\bar{\alpha}_i = \alpha_i/\beta_i \in \overline{K}$ (Ore quotient); if μ is the (nonzero) least common multiple of the β_i, $\sum v_i\bar{\alpha}_i\mu = 0$, and $\bar{\alpha}_i\mu \in K$ by the Ore theory referred to. Hence $\bar{\alpha}_i\mu = 0$, and $\bar{\alpha}_i = 0$, $i = 1, \cdots, n$.

It is trivial to verify that:

(1) $\mathfrak{A} \geqq \mathfrak{B}$ implies $\mathfrak{A}^* \geqq \mathfrak{B}^*$.

(2) $(\mathfrak{A} \cup \mathfrak{B})^* = \mathfrak{A}^* \cup \mathfrak{B}^*$.

(3) $(\mathfrak{A} \cap \mathfrak{B})^* = \mathfrak{A}^* \cap \mathfrak{B}^*$.

For example, in (2) $(\mathfrak{A} \cup \mathfrak{B})^* \geqq \mathfrak{A}^* \cup \mathfrak{B}^*$ follows from (1). But every element in $(\mathfrak{A} \cup \mathfrak{B})^*$ is a $\overline{K}$-form in a K-basis of $\mathfrak{A} \cup \mathfrak{B}$, hence is in $\mathfrak{A}^* \cup \mathfrak{B}^*$. Since $b(\mathfrak{A}^*)$ is the dimension of $\mathfrak{A}^*$ over $\overline{K}$, it follows that $b(\mathfrak{A})$ is a positive modular functional on L.

We may now apply the theory of such functionals [**1**, p. 42, Theorem 3.10] to show that $\delta(\mathfrak{A}, \mathfrak{B}) = b(\mathfrak{A} \cup \mathfrak{B}) - b(\mathfrak{A} \cap \mathfrak{B})$ is a quasi-metric on L:

(4) $\delta(\mathfrak{A}, \mathfrak{B}) \geqq 0$, $\delta(\mathfrak{A}, \mathfrak{A}) = 0$.

(5) $\delta(\mathfrak{A}, \mathfrak{B}) + \delta(\mathfrak{B}, \mathfrak{C}) \geqq \delta(\mathfrak{A}, \mathfrak{C})$.

The relation $\mathfrak{A} \sim \mathfrak{B}$ defined by $\delta(\mathfrak{A}, \mathfrak{B}) = 0$ is an equivalence relation, and the correspondence $\mathfrak{A} \rightarrow [\mathfrak{A}]$, the equivalence class containing $\mathfrak{A}$, is a lattice homomorphism of L onto the metric lattice L' of equivalence classes. For want of a name, we call L' the metric homomorph of L. However, in the correspondence (γ), $\mathfrak{A}^* = \mathfrak{B}^*$ if and only if $\mathfrak{A} \sim \mathfrak{B}$. For, if $\mathfrak{A} \sim \mathfrak{B}$, $b(\mathfrak{A} \cup \mathfrak{B}) = b(\mathfrak{A} \cap \mathfrak{B})$, and $\mathfrak{A}^* \cup \mathfrak{B}^* = \mathfrak{A}^* = \mathfrak{B}^* = \mathfrak{A}^* \cap \mathfrak{B}^*$, since all these have the same dimension over $\overline{K}$. Conversely, if $\mathfrak{A}^* = \mathfrak{B}^*$, then $(\mathfrak{A} \cup \mathfrak{B})^* = \mathfrak{A}^* = (\mathfrak{A} \cap \mathfrak{B})^*$, $b(\mathfrak{A} \cup \mathfrak{B}) = b(\mathfrak{A} \cap \mathfrak{B})$ and $\mathfrak{A} \sim \mathfrak{B}$.

(H) *If K is a right principal ideal domain of integrity, quotient field*

[3] For the elementary divisor theory of matrices over such domains, and references to the literature, see [**2**].

K, then the basis number $b(\mathfrak{N})$ is a positive modular functional on the lattice L of submoduls of $\mathfrak{M}=u_1K+\cdots+u_mK$, and the metric homomorph L' of L is lattice isomorphic with the lattice of submoduls of $\mathfrak{M}^*=u_1\overline{K}+\cdots+u_m\overline{K}$.

4. **Vector spaces over quasi-fields.** We now typify vector spaces over quasi-fields by (I) and (J).

REMARK. *A ring $K=\{0, \alpha, \cdots\}$ with unit ϵ, whose only right ideal $\mathfrak{r}>0$ is K, is a quasi-field.*

Let $\alpha\neq 0$. Then $0<\alpha K=K$, $\alpha\beta=\epsilon$. The right annihilator (right) ideal $\mathfrak{r}$ of α is (0), for $\mathfrak{r}>0$ implies $\mathfrak{r}=K$, and $\alpha\epsilon=\alpha=0$. Hence $\alpha(\beta\alpha-\epsilon)=\alpha\beta\alpha-\alpha=\alpha-\alpha=0$ and $\beta\alpha=\epsilon$.

(I) *Let $\mathfrak{M}=u_1K+\cdots+u_mK$ be a vector space. Then every submodul $\mathfrak{N}$, $0<\mathfrak{N}\leqq\mathfrak{M}$, has a basis of $n\leqq m$ elements, with $\mathfrak{N}<\mathfrak{M}$ implying $n<m$, if and only if K is a quasi-field; that is, the modular functional $b(\mathfrak{N})$ on a vector space over a ring K of the type in* (F) *is sharply positive* [1, p. 41] *if and only if K is a quasi-field.*

These are well known properties of a vector space over a quasi-field. If they hold, then by Lemma 1, the existence of a right ideal $\mathfrak{r}$, $0<\mathfrak{r}<K$ implies $\mathfrak{N}=u_1\mathfrak{r}\cup\cdots\cup u_m\mathfrak{r}<\mathfrak{M}$ with $b(\mathfrak{N})=b(\mathfrak{M})$, contrary to hypothesis. Hence (I) follows from the remark above.

(J) *Let $\mathfrak{M}$ be a vector space over a ring K of the type in* (F). *Then $\mathfrak{M}$ satisfies the descending chain condition if and only if K is a quasi-field.*

For rings of this type, the descending chain condition in $\mathfrak{M}$ and sharp positiveness of $b(\mathfrak{N})$ are equivalent. If $\mathfrak{A}<\mathfrak{B}$ with $b(\mathfrak{A})=b(\mathfrak{B})$, the transformation of $\mathfrak{B}$-basis into $\mathfrak{A}$-basis is of type (iii), on $\mathfrak{B}$.

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