quence of polynomials whose roots lie on the axis of pure imaginaries and which converges uniformly in every finite region.

HUNTER COLLEGE

GENERALIZED LAPLACE INTEGRALS

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We consider the linear space $\mathfrak{H}(c)$ whose elements are functions $f(z)$ $[z=x+iy]$ which are analytic for $x>c$ and satisfy

$$(1) \qquad \int_{-\infty}^{\infty} |f(x+iy)|^2 dy \leqq M, \qquad x>c,$$

where the finite number M depends on the function in question. It is well known that an element $f(z)$ of $\mathfrak{H}(c)$ has boundary values $f(c+iy)$ almost everywhere on $x=c$, and that $\mathfrak{H}(c)$ is a Hilbert space if the norm of $f(z)$ is defined by

$$\|f(z)\|^2 = \int_{-\infty}^{\infty} |f(c+iy)|^2 dy.$$

Furthermore, it is known [5, p. 8] that if $f(z)\in\mathfrak{H}(c)$, then $f(z)$ is representable as a Laplace integral for $x>c$, in the sense that there is a unique function[1] $\phi(t)$ with $e^{-ct}\phi(t)\in L^2(0, \infty)$ such that

$$(2) \qquad \lim_{T\to\infty} \left\| f(z) - \int_0^T e^{-zt}\phi(t)dt \right\| = 0;$$

we shall express (2) by writing

$$(3) \qquad f(z) = \int_0^{\infty} e^{-zt}\phi(t)dt, \qquad x>c.$$

It is easily verified that the integral in (3) converges in the ordinary sense for $x>c$. A Laplace integral may be regarded as a generalized power series; the object of this note is to generalize the integral representation (3) by replacing e^{-zt} by a kernel $g(z, t)$ which is in some sense "nearly" e^{-zt}, just as power series $\sum a_n z^n$ have been generalized[2] by replacing the functions z^n by functions $g_n(z)$.

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[1] Unique, that is, up to sets of measure zero.

[2] For a bibliography of this problem, see [1].

We suppose that $g(z, t)$ is, for each fixed t in $0<t<\infty$, an analytic function of z in $x>c$; and that for each fixed z in $x>c$, $g(z, t)\in L^2(0, R)$ for every $R>0$.

THEOREM 1. *If for each positive* T

$$e^{-ct}\phi(t) \in L^2(0, T) \text{ implies } \int_0^T g(z, t)\phi(t)dt \in \mathfrak{H}(c); \tag{4}$$

and if there is a number λ, $0<\lambda<1$, *such that*

$$\left\| \int_R^S a(t)\left[e^{-zt} - g(z, t)\right]dt \right\| \leqq \lambda \left\| \int_R^S a(t)e^{-zt}dt \right\| \tag{5}$$

for all $S>R>0$ *and all functions* $a(t)\in L^2(R, S)$, *then there exists for each* $f(z)\in\mathfrak{H}(c)$ *a unique* $\psi(t)$ *with* $e^{-ct}\psi(t)\in L^2(0, \infty)$ *such that*

$$f(z) = \int_0^\infty g(z, t)\psi(t)dt, \qquad x > c, \tag{6}$$

where the integral is a mean-square limit for $x\geqq c$, *and also converges in the ordinary sense for* $x>c$.

In Theorems 2, 3 and 5 we shall replace the conditions of Theorem 1 by more convenient conditions; in Theorem 6 the theory will be applied to the generalized Laplace integrals recently discussed by Meijer [4] and Greenwood [7], namely

$$f(z) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty K_\nu(zt)(zt)^{1/2}\phi(t)dt,$$

where $K_\nu(z)$ is the usual notation for a Bessel function of imaginary argument [6, p. 78], and $-\frac{1}{2}<\Re(\nu)<\frac{1}{2}$ (if $\nu=\pm\frac{1}{2}$, we have (3) again).

Theorem 1 is easily proved by the method of successive approximations used for a similar purpose by Paley and Wiener [5, p. 100].

Let $f(z)\in\mathfrak{H}(c)$. Then there is a function $\phi(t)$ with $e^{-ct}\phi(t)\in L^2(0, \infty)$ such that

$$f(z) = \int_0^\infty e^{-zt}\phi(t)dt, \qquad x > c;$$

here, and throughout the proof, integrals over $(0, \infty)$ are taken as mean-square limits, as in (2). Then the integral

$$\int_0^\infty g(z, t)\phi(t)dt, \qquad x > c,$$

exists; for, by (5),

$$\left\| \int_R^S \phi(t)\left[e^{-zt} - g(z, t)\right]dt \right\| \leqq \lambda \left\| \int_R^S \phi(t)e^{-zt}dt \right\| \to 0$$

as R and $S \to \infty$.

We now define inductively sequences $\{\phi_n(t)\}$ and $\{f_n(z)\}$ by setting $\phi_0(t) = \phi(t)$;

$$f_1(z) = \int_0^\infty \phi(t)\left[e^{-zt} - g(z, t)\right]dt = \int_0^\infty \phi_1(t)e^{-zt}dt;$$

and generally

$$f_n(z) = \int_0^\infty \phi_n(t)e^{-zt}dt,$$

$$f_{n+1}(z) = \int_0^\infty \phi_n(t)\left[e^{-zt} - g(z, t)\right]dt.$$

Thus

$$f(z) - f_{n+1}(z) = \int_0^\infty \sum_{k=0}^n \phi_k(t)g(z, t)dt;$$

and by (5)

$$\begin{aligned} \|f_{n+1}\| &\leqq \lambda \left\| \int_0^\infty \phi_n(t)e^{-zt}dt \right\| = \lambda \|f_n\| \\ &\leqq \cdots \leqq \lambda^{n+1}\|f\| \to 0, \qquad n \to \infty. \end{aligned} \tag{7}$$

Consequently

$$\left\| f(z) - \int_0^\infty \sum_{k=0}^n \phi_k(t)g(z, t)dt \right\| \to 0, \qquad n \to \infty. \tag{8}$$

Now

$$\left\| \int_0^\infty \phi_n(t)e^{-zt}dt \right\| = \|f_n\| \leqq \lambda^{n+1}\|f\|,$$

and hence

$$\left\| \int_0^\infty \sum_m^n \phi_k(t)e^{-zt}dt \right\| \leqq \sum_m^n \lambda^{k+1}\|f\| \to 0, \qquad m, n \to \infty.$$

Therefore, since $\mathfrak{H}(c)$ is complete, there exists $F(z) \in \mathfrak{H}(c)$ such that

$$\left\| F(z) - \int_0^\infty \sum_{k=0}^n \phi_k(t)e^{-zt}dt \right\| \to 0, \qquad n \to \infty, \tag{9}$$

and

$$F(z) = \int_0^\infty \psi(t)e^{-zt}dt, \qquad e^{-ct}\psi(t) \in L^2(0, \infty).$$

If we set

$$\psi_n(t) = \psi(t) - \sum_{k=0}^{n} \phi_k(t) \tag{10}$$

we then have

$$\left\| \int_0^\infty \psi_n(t) e^{-zt} dt \right\| \to 0;$$

then by (5)

$$\left\| \int_0^\infty g(z, t) \psi_n(t) dt \right\| \leqq (1 + \lambda) \left\| \int_0^\infty e^{-zt} \psi_n(t) dt \right\| \to 0.$$

By (10) and (8) we thus have

$$f(z) = \int_0^\infty \psi(t) g(z, t) dt,$$

and the existence of the representation (6) is established. That (6) converges in the ordinary sense for $x > c$ follows easily from the fact that functions of $\mathfrak{H}(c)$ are represented by their Cauchy integrals [**2**, p. 338].

To show that the representation is unique, we have only to show that

$$\int_0^\infty \omega(t) g(z, t) dt \equiv 0, \qquad x > c, \tag{11}$$

implies $\omega(t) = 0$ almost everywhere. Now if (11) is true, for every positive T we have, by (5),

$$\begin{aligned} \left\| \int_0^T \omega(t) e^{-zt} dt \right\| &\leqq \left\| \int_0^T \omega(t) \left[e^{-zt} - g(z, t) \right] dt \right\| + \left\| \int_0^T \omega(t) g(z, t) dt \right\| \\ &\leqq \lambda \left\| \int_0^T \omega(t) e^{-zt} dt \right\| + \left\| \int_0^T \omega(t) g(z, t) dt \right\|. \end{aligned}$$

The limit of the last term on the right is zero, and since $\lambda < 1$ it follows that

$$\int_0^\infty \omega(t) e^{-zt} dt = 0, \qquad x > c.$$

Since the Laplace representation (3) is unique, $\omega(t) = 0$ almost everywhere.

THEOREM 2. *If for* $x > c$

$$g(z, t) = e^{-zt} \{ 1 + h(z, t) \} \tag{12}$$

with

$$h(z, t) = \int_0^\infty e^{-zu}\omega(t, u)du, \qquad x > c,$$

$$| \omega(t, u) | \leqq \omega(u),$$

$$h(z) = \int_0^\infty e^{-zu}\omega(u)du, \qquad x > c,$$

and

$$h(c) < 1, \tag{13}$$

then the function $g(z, t)$ *has properties* (4) *and* (5), *and consequently the representation* (6) *is possible and unique.*

If $a(t) \in L^2(R, S)$, we set $a^*(t) = a(t)$ in (R, S) and $a^*(t) = 0$ outside (R, S), and use the Parseval theorem for Laplace transforms; with $z = c + iy$,

$$\begin{aligned}
\int_{-\infty}^{\infty} \left| \int_R^S e^{-zt}a(t)h(z, t) \right|^2 dy
&= \int_{-\infty}^{\infty} \left| \int_0^\infty e^{-zt}a^*(t)dt \int_0^\infty e^{-zu}\omega(t, u)du \right|^2 dy \\
&= \int_{-\infty}^{\infty} \left| \int_0^\infty e^{-zv}dv \int_0^v a^*(t)\omega(t, v - t)dt \right|^2 dy \\
&= \frac{1}{2\pi}\int_0^\infty \left| e^{-cv} \int_0^v a^*(t)\omega(t, v - t)dt \right|^2 dv \\
&\leqq \frac{1}{2\pi}\int_0^\infty \left| e^{-cv} \int_0^v | a^*(t) | \omega(v - t)dt \right|^2 dv \\
&= \int_{-\infty}^{\infty} \left| \int_0^\infty e^{-zv}dv \int_0^v | a^*(t) | \omega(v - t)dt \right|^2 dy \\
&= \int_{-\infty}^{\infty} \left| \int_0^\infty e^{-zt} | a^*(t) | dt \int_0^\infty e^{-zu}\omega(u)du \right|^2 dy \\
&= \int_{-\infty}^{\infty} \left| h(z) \int_0^\infty e^{-zt} | a^*(t) | dt \right|^2 dy \\
&\leqq [h(c)]^2 \int_{-\infty}^{\infty} \left| \int_0^\infty e^{-zt} | a^*(t) | dt \right|^2 dy \\
&= [h(c)]^2 \int_{-\infty}^{\infty} \left| \int_R^S e^{-zt}a(t)dt \right|^2 dy.
\end{aligned}$$

Since $h(c)<1$, this gives us (5). But it also implies

$$\left\{\int_{-\infty}^{\infty}\left|\int_0^T \phi(t)g(z,t)dt\right|^2 dy\right\}^{1/2}$$
$$\leqq \{1+h(c)\}\left\{\int_{-\infty}^{\infty}\left|\int_0^T \phi(t)e^{-zt}dt\right|^2 dy\right\}^{1/2}$$
$$=\frac{1}{2\pi}\{1+h(c)\}\left\{\int_0^T |\phi(t)|^2 e^{-2ct}dt\right\}^{1/2},$$

and hence (4).

For example, (13) is satisfied, with $c=1$, if $h(z,t)=z/(z^2+t^2)$ or $1/(z+t)$.

THEOREM 3. *If $h(z,t)$ is defined by* (12) *and satisfies*

$$|h(z,t)| \leqq |k(z)|\, l(t), \qquad x>c, \tag{14}$$

with $\|k(z)\|(\int_0^\infty |l(t)|^2dt)^{1/2}=\mu<(2\pi)^{-1/2}$, *then $g(z,t)$ has properties* (4) *and* (5) *and the representation* (6) *is possible.*

In fact, we have

$$\left|\int_R^S a(t)e^{-zt}h(z,t)dt\right| \leqq |k(z)|\int_R^S e^{-ct}l(t)\,|a(t)|\,dt,$$

$$\left\|\int_R^S a(t)e^{-zt}h(z,t)dt\right\|$$
$$\leqq \|k(z)\|\left\{\int_R^S |l(t)|^2dt\right\}^{1/2}\left\{\int_R^S e^{-2ct}|a(t)|^2dt\right\}^{1/2}$$
$$\leqq (2\pi)^{1/2}\mu\left\|\int_R^S a(t)e^{-zt}dt\right\|.$$

This verifies (5); and (4) follows because

$$\left\|\int_0^T a(t)g(z,t)dt\right\| = \left\|\int_0^T a(t)e^{-zt}[1+h(z,t)]dt\right\|$$
$$\leqq \left\|\int_0^T a(t)e^{-zt}dt\right\| + \left\|\int_0^T a(t)e^{-zt}h(z,t)dt\right\|$$
$$\leqq [1+(2\pi)^{1/2}\mu]\left\|\int_0^\infty a(t)e^{-zt}dt\right\|.$$

THEOREM 4. *If $g(z, t) = g^*(z, t)$ for $t \geqq s > 0$; if the representation* (6) *is possible and unique for $x > c^*$, with $g(z, t)$, for every $f(z)$ of $\mathfrak{H}(c^*)$; and possible and unique for $x > c$, with $g^*(z, t)$, for every $f(z)$ of $\mathfrak{H}(c)$ (where $c < c^*$), then the representation is also possible and unique for $x > c$, with $g(t, z)$, for every $f(z)$ of $\mathfrak{H}(c)$.*

We have

$$f(z) = \int_0^\infty g(z, t)\psi(t)dt, \qquad x > c^*. \tag{15}$$

Define

$$F(z) = f(z) - \int_0^s g(z, t)\psi(t)dt = \int_s^\infty g(z, t)\psi(t)dt = \int_s^\infty g^*(z, t)\psi(t)dt.$$

The last integral defines $F(z)$ for $x > c$; and since the representation of $F(z)$ in terms of $g^*(z, t)$ is unique for $x > c$, the integral

$$\int_s^\infty g(z, t)\psi(t)dt$$

must represent $F(z)$ for $x > c$; hence (15) represents $f(z)$ for $x > c$ also.

THEOREM 5. *Theorem 3 remains true if* (14) *is satisfied with $k(z) \in \mathfrak{H}(c)$ and $l(t) \in L^2(0, \infty)$.*

If $k(z) \in \mathfrak{H}(c)$ we have

$$\lim_{x\to\infty} \int_{-\infty}^\infty | k(x + iy) |^2 dy = 0; \tag{16}$$

for, if $x > c$,

$$k(x + iy) = \int_0^\infty e^{-xt-iyt}\phi(t)dt, \qquad e^{-ct}\phi(t) \in L^2(0, \infty);$$

and so

$$\int_{-\infty}^\infty | k(x + iy) |^2 dy = \frac{1}{2\pi}\int_0^\infty e^{-2xt} | \phi(t) |^2 dt \to 0, \qquad x \to \infty.$$

Choose s so large that

$$\| k(z) \| \left(\int_s^\infty | l(t) |^2 dt \right)^{1/2} = \mu < (2\pi)^{-1/2}. \tag{17}$$

Then define $h^*(z, t) = h(z, t)$ for $t > s$, $h^*(z, t) = 0$ for $0 \leqq t \leqq s$; and

$g^*(z, t) = e^{-zt}[1+h^*(z, t)]$. The function $g^*(z, t)$ satisfies the conditions of Theorem 3, with $l(t)$ replaced by the function $l^*(t)$ equal to $l(t)$ in (s, ∞) and to zero in $(0, s)$. On the other hand, if b is so large that

$$\left\{\int_{-\infty}^{\infty} |k(b+iy)|^2 dy\right\}^{1/2}\left\{\int_0^{\infty} |l(t)|^2 dt\right\}^{1/2} < (2\pi)^{-1/2}$$

(such a b exists because of (16)), the function $g(z, t)$ satisfies the conditions of Theorem 3 with c replaced by b. Thus the representation (6) is possible and unique with $g(z, t)$ for $x > b$, and with $g^*(z, t)$ for $x > c$. Theorem 5 now follows from Theorem 4.

Let $K_\nu(z)$ have its usual meaning in the theory of Bessel functions [6, p. 78].

THEOREM 6.[3] *If* $-\frac{1}{2} < \Re(\nu) < \frac{1}{2}$, *then for every* $f(z) \in \mathfrak{H}(0)$ *there is a unique* $\psi(t) \in L^2(0, \infty)$ *such that*

$$f(z) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} K_\nu(zt)(zt)^{1/2}\psi(t)\,dt, \qquad x > 0, \tag{18}$$

where the integral is a mean-square limit, as in (2), *for* $x \geqq 0$, *and converges in the ordinary sense for* $x > 0$.

We shall show that the function $(2zt/\pi)^{1/2}K_\nu(zt)$ satisfies the hypotheses of Theorem 5, with $c = 0$. The following inequalities for $K_\nu(z)$ are valid for $\Re(z) > 0$ [6, p. 219; 3, p. 658].

$$(2z/\pi)^{1/2}e^z K_\nu(z) = R_0(z), \tag{19}$$

$$(2z/\pi)^{1/2}e^z K_\nu(z) = 1 + R_1(z)/(2z), \tag{20}$$

where

$$|R_j(z)| \leqq \left|\frac{\cos \nu\pi}{\cos \Re(\nu\pi)}\right|, \qquad j = 0, 1.$$

If now $h(z, t)$ is defined by

$$(2zt/\pi)^{1/2}K_\nu(zt) = e^{-zt}[1 + h(z, t)],$$

from (20) and (19) we have, with $A(\nu)$ depending only on ν,

$$|h(z, t)| \leqq \frac{A(\nu)}{|zt|}, \qquad |h(z, t)| \leqq A(\nu), \qquad x > 0,\ 0 < t < \infty.$$

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[3] A closely related theorem is given by Meijer [4, p. 603]. Meijer also obtains an inversion formula for (18).

Hence, with some $B(\nu)$,

$$| h(z, t) | \leqq \frac{B(\nu)}{| 1 + z | (1 + t)}, \quad x > 0, 0 < t < \infty,$$

and the conditions of Theorem 5 are satisfied.

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