$$x_1 = A_{1,\rho}(u, v)$$
$$\equiv \frac{a^2 u}{\rho^2(u^2+v^2)} \left[a^2+u^2+v^2+\rho^2-((a^2+\rho^2-u^2-v^2)^2+4a^2(u^2+v^2))^{1/2}\right],$$
$$x_2 = A_{2,\rho}(u, v)$$
$$\equiv \frac{a^2 v}{\rho^2(u^2+v^2)} \left[a^2+u^2+v^2+\rho^2-((a^2+\rho^2-u^2-v^2)^2+4a^2(u^2+v^2))^{1/2}\right],$$
$$x_3 = A_{3,\rho}(u, v)$$
$$\equiv a - \frac{2a^3}{\rho^2} \log\left[\frac{a^2-u^2-v^2+\rho^2+((a^2+\rho^2-u^2-v^2)^2+4a^2(u^2+v^2))^{1/2}}{2a^2}\right].$$

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NOTE ON THE DISTRIBUTION OF VALUES OF THE ARITHMETIC FUNCTION $d(m)$[1]

M. KAC

1. **Introduction.** Recently Dr. Erdös and the present writer[2] proved the following theorem:

If $\nu(m)$ denotes the number of different prime divisors of m and $k_n(\omega)$ the number of positive integers $m \leqq n$ for which

$$\nu(m) \leqq \lg \lg n + \omega(2 \lg \lg n)^{1/2},$$

then

$$\lim_{n\to\infty} \frac{k_n(\omega)}{n} = \pi^{-1/2} \int_{-\infty}^{\omega} e^{-u^2} du = D(\omega).$$

The purpose of this note is to derive a similar theorem concerning the function $d(m)$ which denotes the number of all different divisors of m (1 and m are included).

In fact we are going to prove the following theorem:

If $r_n(\omega)$ denotes the number of positive integers $m \leqq n$ for which

$$d(m) \leqq 2^{\lg \lg n + \omega(2 \lg \lg n)^{1/2}},$$

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[1] Presented to the Society, May 2, 1941.

[2] P. Erdös and M. Kac, *The Gaussian law of errors in the theory of additive number theoretic functions*, American Journal of Mathematics, vol. 62, pp. 738–742.

then

$$\lim_{n\to\infty} \frac{r_n(\omega)}{n} = \pi^{-1/2} \int_{-\infty}^{\omega} e^{-u^2} du = D(\omega).$$

2. **Proof of the theorem.** The proof is based on the theorem cited in the introduction and on the following two facts:

I. The mean value

$$M\{d(m)/2^{\nu(m)}\} = \lim_{n\to\infty} n^{-1} \sum_{m=1}^{n} d(m)/2^{\nu(m)}$$

exists and is finite.

II. If $f(m) \geqq 0$ is such that $M\{f(m)\}$ is finite, if $\lim g(n) = \infty$ as $n \to \infty$ and if $p(n)$ denotes the number of positive integers $m \leqq n$ for which $f(m) \leqq g(n)$, then $\lim p(n)/n = 1$ as $n \to \infty$.

I is implied by a theorem of E. R. van Kampen and A. Wintner[3] and II is almost obvious even under a weaker condition that $\limsup n^{-1}\sum_1^n f(m) < \infty$.

Let ω be an arbitrary real number and $\epsilon > 0$. Put $f(m) = d(m)/2^{\nu(m)}$ and $g(n) = 2^{\epsilon(2 \lg \lg n)^{1/2}}$. Let F_n be the set of positive integers $m \leqq n$ for which $\nu(m) \leqq \lg \lg n + (\omega - \epsilon)(2 \lg \lg n)^{1/2}$, G_n the set of positive integers $m \leqq n$ for which $f(m) \leqq g(n)$ and H_n the set of positive integers $m \leqq n$ for which $d(m) \leqq 2^{\lg \lg n + \omega(2 \lg \lg n)^{1/2}}$. If $m \in F_n G_n$ then $m \in H_n$. Hence,

$$F_n G_n \subset H_n.$$

The number of elements in F_n is $k_n(\omega - \epsilon)$; in G_n, $p(n)$; and in H_n, $r_n(\omega)$.

Thus, the number of elements in $F_n G_n$ is $\geqq k_n(\omega - \epsilon) - (n - p(n))$ and finally

$$k_n(\omega - \epsilon) - (n - p(n)) \leqq r_n(\omega).$$

On the other hand for every m, $2^{\nu(m)} \leqq d(m)$ (the equality occurs only if m is a prime) and therefore $H_n \subset F_n$ or $r_n(\omega) \leqq k_n(\omega)$. The inequalities combined give

$$k_n(\omega - \epsilon) - (n - p(n)) \leqq r_n(\omega) \leqq k_n(\omega).$$

But as $n \to \infty$ $k_n(\omega - \epsilon)/n \to D(\omega - \epsilon)$, $k_n(\omega)/n \to D(\omega)$ and $(n - p(n))/n \to 0$ (see I and II); hence

$$D(\omega - \epsilon) \leqq \liminf_{n\to\infty} \frac{r_n(\omega)}{n} \leqq \limsup_{n\to\infty} \frac{r_n(\omega)}{n} \leqq D(\omega).$$

[3] American Journal of Mathematics, vol. 62, p. 618 (Theorem IV).

Since ϵ is arbitrary and $D(\omega)$ is a continuous function of ω,

$$\lim_{n\to\infty} \frac{r_n(\omega)}{n} = D(\omega).$$

3. **Some numerical results.** The values of $d(m)$ for $1 \leqq m \leqq 10^4$ can be found in the recent tables of the British Association for the Advancement of Science,[4] so that it was easy to obtain the exact value of $10^{-4}r_{10^4}(\omega)$ for different values of ω and compare them with the values of $D(\omega)$ computed from the tables of the probability integral.[5]

$2^{\lg \lg n+\omega(2 \lg \lg n)^{1/2}}$	ω	$10^{-4}r_{10^4}(\omega)$	$D(\omega)$
1	−1.054	0.0001	0.0680
2	−0.579	0.1230	0.3065
3	−0.302	0.1255	0.4346
4	−0.105	0.3863	0.4410
5	0.048	0.3867	0.5220
6	0.173	0.4631	0.5961
7	0.279	0.4633	0.6534
8	0.370	0.6747	0.6996
9	0.451	0.6779	0.7382
10	0.523	0.6929	0.7702
11	0.588	0.7970	0.7971
12	0.648	0.7971	0.8202
13	0.702	0.8012	0.8396
14	0.753	0.8027	0.8565
15	0.800	0.8827	0.8710
16	0.845	0.8827	0.8840

For a good agreement $n = 10{,}000$ seems to be too small. The rather striking fact that $10^{-4}r_{10^4}(.588)$ is almost equal to $D(.588)$ is probably accidental. The case $\omega = .800$ disproves the conjecture that always $n^{-1}r_n(\omega) \leqq D(\omega)$.

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[4] *Mathematical Tables*, vol. 8, *Number-Divisor Tables*, Cambridge University Press, 1940. We refer in particular to Table III. In these tables $d(m)$ is denoted by $\nu(m)$.

[5] We used the tables on pages 388 to 391 of the first volume of Czuber's *Wahrscheinlichkeitsrechnung*, Teubner, 1908. It should be noted that $D(\omega) = \{1 + \Phi(\omega)\}/2$.

We also wish to thank Mr. W. J. Harrington for his help in computing the table given on this page.