ON THE EXISTENCE OF ELECTRICAL NETWORKS

C. E. CLARK

This paper is concerned with the existence of electrical networks which satisfy certain preassigned conditions. These conditions have to do with the existence of circuits with preassigned resistances in common.

Consider a finite set of points in euclidean 3-space and a set of straight line segments joining pairs of these points. Furthermore, suppose that no two of the segments intersect at an interior point, but any number of segments may have a common end-point. Each of the segments is called a *branch* of the graph. With each branch of the graph let there be associated a non-negative real number called the *resistance* of the branch. The graph together with the resistances is an *electrical network*. A *circuit* of a network is a topological circle of the network together with an orientation of this circle.

Let two circuits C_i and C_j have the branches b_{ij}^p, $p=1, 2, \cdots$, in common. Let r_{ij}^p be the resistance of b_{ij}^p. Let $\bar{r}_{ij}^p=r_{ij}^p$ if the orientations of C_i and C_j agree along b_{ij}^p while $\bar{r}_{ij}^p=-r_{ij}^p$ if the orientations are opposite. Then $I_{ij}=\sum_p \bar{r}_{ij}^p$, $p=1, 2, \cdots$, is the *intersection* of C_i and C_j. We see that I_{ii} is the sum of the resistances of the branches of C_i.

Let C_i, $i=1, 2, \cdots, n$, be distinct circuits of a network. Then the matrix $\|I_{ij}\|$, $i, j=1, \cdots, n$, is the *intersection matrix of the* C_i. This matrix is symmetric and has non-negative diagonal elements. Any matrix with these two properties is an *intersection matrix*. An intersection matrix M is *realizable* when there exists a network which has a set of circuits whose intersection matrix is M.

THEOREM 1. *Given an intersection matrix*

$$\|I_{ij}\|, \qquad i, j = 1, \cdots, n; \tag{1}$$

if $I_{ii}\geqq\sum_j |I_{ij}|$, $j\neq i$, $i=1, \cdots, n$, *then* (1) *is realizable.*

PROOF. Choose $2n$ points on each of n oriented circles S_i, $i=1, \cdots, n$. These points divide each S_i into $2n$ branches. Denote the $2n$ branches of S_i by $b_{i1}, o_{i1}, b_{i2}, o_{i2}, \cdots, b_{in}, o_{in}$ with the b's separated by the o's. To each o is assigned the resistance zero. We shall next identify b_{ij} and b_{ji}, $i\neq j$.[1] In the resulting figure there will be a circle that is the natural topological image of S_i. We shall say that this circle *is* S_i.

[1] I.e., we bring the two end-points of b_{ij} into coincidence with the end-points of b_{ji} and replace these two branches by a single branch. To do this it may be necessary to replace any figure by its homeomorph.

The identification of b_{ij} and b_{ji} can be performed in two ways. If $I_{ij}>0$, the identification is made so that S_i and S_j agree in orientation along their common branch; if $I_{ij}<0$, S_i and S_j have opposite orientations along their common branch; and if $I_{ij}=0$, we choose either identification. To the common branch is assigned the resistance $|I_{ij}|$.

To b_{ii}, $i=1, \cdots, n$, is assigned the resistance $I_{ii}-\sum_j|I_{ij}|$, $j\neq i$. In the resulting network the S_i, $i=1, \cdots, n$, are circuits whose intersection matrix is (1).

THEOREM 2. *If matrix* (1) *is realizable, and if a matrix* (1a) *is obtained from* (1) *by increasing a diagonal term* I_{ii} *to* I_{ii}', *then* (1a) *is realizable.*

PROOF. Consider a realization of (1) and let C_i be the circuit whose total resistance is I_{ii}. We break C_i at one of its vertices A and insert two branches b_i and b between the two ends A_1 and A_2 of the broken circuit. Any other branch that had been connected at A may now be connected at either A_1 or A_2. To the branch b_i is assigned the resistance $I_{ii}'-I_{ii}$, while to b is assigned the resistance zero. The new network gives a realization of (1a).

The above theorems show that if all except the diagonal elements of a symmetric matrix are given, then it is possible to assign the diagonal elements so that the resulting matrix is realizable. Also there are minimal sets of diagonal elements that can be assigned. But there is not a unique minimal set of diagonal elements as we now show by example.

We wish to find the smallest a, b, and c so that

$$\left\|\begin{matrix} a & 1 & 1 \\ 1 & b & -1 \\ 1 & -1 & c \end{matrix}\right\| \tag{2}$$

is realizable. We can take $a=1$. To do so we must have the second and third circuits S_2 and S_3 intersecting the first circuit S_1 along the same branch and both S_2 and S_3 agreeing in orientation with S_1 along this branch. Since $I_{23}=-1$, we must have S_2 and S_3 intersecting along another branch. This branch can have resistance 2 with S_2 and S_3 oppositely oriented here. This gives a realization of (2) with $a=1$, $b=3$, and $c=3$.

Also we can see that if $a<2$, then one other of the diagonal terms must be greater than 2. Hence if $a=b=c=2$ makes (2) realizable, then this set of diagonal elements is a minimal set (in the sense that one element can not be decreased without increasing another). But by Theorem 1 the values $a=b=c=2$ make (2) realizable. Hence there is not a unique minimal set of values for a, b, and c.

THEOREM 3. *Given* I_{ij}, $i\neq j$, *of* (1) (*but not the diagonal elements*), *let* a *be fixed; then we can take* $I_{aa}=\max |I_{aj}|$, $j\neq a$, *and find values for* I_{ii}, $i\neq a$, *so that* (1) *is realizable.*

The proof is similar to the proofs of the preceding theorems.

PURDUE UNIVERSITY

ON THE SPHERICAL SURFACE OF SMALLEST RADIUS ENCLOSING A BOUNDED SUBSET OF n-DIMENSIONAL EUCLIDEAN SPACE[1]

L. M. BLUMENTHAL AND G. E. WAHLIN

1. **Introduction.** An $(n-1)$-dimensional spherical surface $S_{n-1,r}$ is the "surface" of an n-dimensional sphere of radius r in E_n, the n-dimensional euclidean space. A given spherical surface encloses M, a subset of E_n, provided M is contained in the sphere with this surface, while M is enclosable by a given $S_{n-1,r}$ whenever M is a subset of a sphere whose surface is congruent with $S_{n-1,r}$. The purpose of this article is to show (1) *if* M *is any bounded subset of* E_n (*containing more than a single point*) *there exists a unique* $S_{n-1,r}$ *of smallest radius* r *enclosing* M and (2) *if* d *is the diameter of* M, *then the radius of the unique smallest* $S_{n-1,r}$ *enclosing* M *satisfies the relation* $r\leqq [n/2(n+1)]^{1/2}\cdot d$.

In a proof that abounds with algebraic difficulties, H. W. E. Jung established these results in his dissertation (1901) for the case of finite point sets and indicated their extension to infinite sets at the end of his long paper.[2] Returning to the subject eight years later, Jung attempted a geometric proof for the case of *n points in a plane*, but succeeded in obtaining in this later article only necessary conditions on the smallest circle enclosing a plane (finite) set, since his procedure yields the smallest circle only in case one is assured of the existence of such a circle.[3] Though this fact can readily be supplied, the geometric considerations used by Jung are not easily extended to finite

[1] Presented to the Society, February 22, 1941.

[2] H. W. E. Jung, *Ueber die kleinste Kugel, die eine räumliche Figur einschliesst*, Journal für die reine und angewandte Mathematik, vol. 123 (1901), pp. 241–257.

[3] H. W. E. Jung, *Ueber den kleinsten Kreis, der eine ebene Figur einschliesst*, ibid., vol. 137 (1909), pp. 310–313.