ON THE SIMULTANEOUS APPROXIMATION OF TWO REAL NUMBERS[1]

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If $\xi_1, \xi_2, \cdots, \xi_n$ are any real numbers and t is a positive integer, then it is well known that integers $a_1, a_2, \cdots, a_n, b$ can be found, such that $0<b\leqq t^n$ and

$$|b\xi_k - a_k| < 1/t, \qquad k = 1, 2, \cdots, n.$$

The proof is briefly the following.[2] Consider the t^n+1 points $(r\xi_1, r\xi_2, \cdots, r\xi_n)$, where $r=0, 1, \cdots, t^n$. Reduce mod 1 to congruent points in the unit cube $(0\leqq x_1<1, \cdots, 0\leqq x_n<1)$. If this cube is divided into t^n cubes of edge $1/t$ (including the lower boundaries), then at least one of these small cubes must contain two of the reduced points, say those with $r=r'$ and $r=r''$. With $b=|r'-r''|$ and suitable a's, we evidently satisfy the required inequalities.

For $n=1$, the inequality can be sharpened to

$$|b\xi - a| \leqq 1/(t+1),$$

b satisfying the condition $0<b\leqq t$. For if we consider the points $r\xi$ $(r=0, 1, \cdots, t)$, and mark the points in the interval $0\leqq x\leqq 1$ which are congruent to them mod 1, we have at least $t+2$ points marked, since corresponding to $r=0$ we mark both 0 and 1. Some two of the marked points must lie within a distance $1/(t+1)$ from each other, so that the desired conclusion follows. This is the best result, as the example $\xi=1/(t+1)$ shows.

The present note solves the corresponding problem for $n=2$. For larger values of n the problem appears more difficult.

THEOREM. *If ξ_1 and ξ_2 are any real numbers, and s is a positive integer, then integers a_1, a_2, b can be found, such that $0<b\leqq s$, and*

$$|b\xi_k - a_k| \leqq \max\left(\frac{[s^{1/2}]}{s+1}, \frac{1}{[s^{1/2}]+1}\right), \qquad k = 1, 2.$$

For every s, values of ξ_1 and ξ_2 can be found for which the inequalities could not both be satisfied if the equality sign were omitted.

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[2] The method used in this proof (*Schubfachprinzip* or "pigeonhole principle") was first used by Dirichlet in connection with a similar problem. We sketch the proof here in order to compare it with the proof of the theorem below, which also uses that method.

The inequalities may also be written

$$| b\xi_k - a_k | \leqq \begin{cases} t/(s+1) & \text{for} \quad t^2 - 1 \leqq s \leqq t(t+1) - 1, \\ 1/(t+1) & \text{for} \quad t(t+1) - 1 \leqq s \leqq (t+1)^2 - 1. \end{cases}$$

It will be noted that in some intervals the bound does not decrease as s increases.

We show first that the theorem is the best possible. We shall think of the inequalities in the form just given. If $s<(t+1)^2$, then it is evident that $\xi_1=1/(t+1)$, $\xi_2=1/(t+1)^2$ are a pair of real numbers which cannot be approximated simultaneously with an error less than $1/(t+1)$; this settles the second case. For the first case, consider the pair of real numbers $\xi_1=1/(s+1)$, $\xi_2=t/(s+1)$. We are to show that not both errors can be made less than $t/(s+1)$. We note first that $b\xi_1$ and $b\xi_2$ differ from integers by the same amount as $(s+1-b)\xi_1$ and $(s+1-b)\xi_2$; hence we may suppose that $b\leqq(s+1)/2$, and therefore $0<b\xi_1\leqq 1/2$. In order to make $|b\xi_1-a_1|<t/(s+1)$, we must have $0<b<t$. Then $0<b\xi_2<1$. Since $b\xi_2\geqq\xi_2=t/(s+1)$ and $1-b\xi_2\geqq 1-(t-1)\xi_2=1-(t-1)t/(s+1)\geqq t/(s+1)$, we see that the inequality $|b\xi_2-a_2|<t/(s+1)$ cannot be satisfied.

The theorem evidently follows from the lemma below, by putting $t=[s^{1/2}]$.

LEMMA. *Let s and t be positive integers with $s\geqq t$. If ξ_1 and ξ_2 are any real numbers, then integers a_1, a_2, b can be found, such that $0<b\leqq s$, and*

$$| b\xi_1 - a_1 | \leqq t/(s+1), \qquad | b\xi_2 - a_2 | \leqq 1/(t+1).$$

PROOF. Consider the points $(r\xi_1, r\xi_2)$ with $r=0, 1, \cdots, s$. Mark all the points congruent to these mod 1 which fall in the rectangle $0\leqq x_1\leqq t$, $0\leqq x_2<1$. There are $(s+1)t$ points to be marked with $x_1<t$; and in addition, the point $(t, 0)$ is marked, corresponding to $r=0$. If we divide our rectangle into $s+1$ rectangles of width $t/(s+1)$ (closed except at the top) by means of vertical lines, then at least one of them contains more than t points, all corresponding to different values of r. The corresponding values of x_2 are $t+1$ or more numbers, some two of which differ mod 1 by not more than $1/(t+1)$. Thus we find two points $(r'\xi_1, r'\xi_2)$ and $(r''\xi_1, r''\xi_2)$, whose horizontal distance mod 1 does not exceed $t/(s+1)$ and whose vertical distance mod 1 does not exceed $1/(t+1)$. Putting $b=|r'-r''|$ gives the required result.

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