ON SPHERICAL CYCLES[1]

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Given a metric separable space Y, we consider the homology group $B^n(Y)$ obtained using n-dimensional singular cycles in Y with integer coefficients. Every continuous mapping $f \in Y^{S^n}$ of the oriented n-dimensional sphere S^n into Y defines uniquely an element $h(f)$ of $B^n(Y)$. Clearly if $f_0, f_1 \in Y^{S^n}$ are two homotopic mappings, then $h(f_0) = h(f_1)$.

The homology classes $h(f)$ will be called *spherical homology classes*. A cycle will be called *spherical* if its homology class is spherical.[2]

THEOREM 1. *If Y is arcwise connected, the spherical homology classes form a subgroup of $B^n(Y)$.*

Let $p \in S^n$, $q \in Y$, and let $S^n = S^n_+ + S^n_-$ be a decomposition of S^n into two hemispheres such that $p \in S^n_+ \cdot S^n_-$. Consider $f_0, f_1 \in Y^{S^n}$. It is well known that, replacing if necessary f_0 and f_1 by homotopic mappings, we may assume that $f_0(S^n_+) = q$ and that $f_1(S^n_-) = q$. Defining $f = f_0$ on S^n_- and $f = f_1$ on S^n_+ we clearly have

$$f \in Y^{S^n}, \qquad h(f) = h(f_0) + h(f_1).$$

The homology class $h(f_0) + h(f_1)$ is therefore spherical.

Let M^r be an r-dimensional (finite or infinite) manifold[3] and P^{r-n-1} $(n > 0)$ an at most $(r-n-1)$-dimensional subpolyhedron of M^r.

THEOREM 2. *Every n-dimensional cycle γ^n in $M^r - P^{r-n-1}$ such that $\gamma^n \sim 0$ in M^r is spherical (with respect to $M^r - P^{r-n-1}$).*

Let a^{r-n-1} be an $(r-n-1)$-dimensional simplex of M^r and b^{n+1} the $(n+1)$-cell dual to it. The boundary ∂b^{n+1} is contained in $M^r - P^{r-n-1}$ and is a spherical cycle. Since $M^r - P^{r-n-1}$ is connected, the spherical homology classes of $B^n(M^r - P^{r-n-1})$ form a group. It follows that each cycle of the form

$$(*) \qquad \partial\left(\sum_i \alpha_i b_i^{n+1}\right)$$

is a spherical cycle with respect to $M^r - P^{r-n-1}$. The cycle γ^n is homologous in $M^r - P^{r-n-1}$ to a cycle of the form (*). Therefore γ^n is spherical.

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[1] Presented to the Society, April 13, 1940.

[2] Spherical cycles were considered by W. Hurewicz, Proceedings, Akademie van Wetenschappen te Amsterdam, vol. 38 (1935), pp. 521–528.

[3] See K. Reidemeister, *Topologie der Polyeder*, Leipzig, 1938, p. 151.

THEOREM 3. *Let γ^n be a spherical cycle in M^r and let $r>2n$. Then there is a simplicial homeomorphism*[4] $g\in M^{r^{S^n}}$ *such that* $\gamma^n\in h(g)$.

This is an immediate consequence of Theorem 5 below. Using Theorem 2 we obtain the following:

THEOREM 4. *Given an n-cycle* $\gamma^n\subset M^r-P^{r-n-1}$ $(r>2n)$ *such that* $\gamma^n\sim 0$ *in* M^r, *there is a cycle* $\gamma_1^n\subset M^r-P^{r-n-1}$ *which is a simplicial and homeomorphic image of* S^n *such that* $\gamma^n\sim\gamma_1^n$ *in* M^r-P^{r-n-1}.

THEOREM 5. *Let Q^n be a finite n-dimensional polyhedron and let $r>2n$. Every continuous mapping $f\in M^{r^{Q^n}}$ can be approached by simplicial homeomorphisms $g\in M^{r^{Q^n}}$.*

We may admit that the mapping f is simplicial. Let $a_1, a_2, \cdots, a_k$ be the vertices of the complex $f(Q^n)$ and let $\sigma_1, \sigma_2, \cdots, \sigma_k$ be the corresponding stars.[5] Let us choose $\delta>0$ so that $x\in f(Q^n)$ will imply $\rho(x, M^r-\sigma_i)>\delta$ for some $i=1, 2, \cdots, k$.

Let $\delta>2\epsilon>0$. We are going to define a sequence $f=f_0, f_1, \cdots, f_k$ of simplicial maps of Q^n into M^r such that

$$(1)\qquad |f_i(x)-f_{i-1}(x)|<\frac{\epsilon}{k},$$

$$(2)\qquad f_i(x_1)=f_i(x_2)\quad\text{implies}\quad f_{i-1}(x_1)=f_{i-1}(x_2),$$

$$(3)\qquad x_1\neq x_2 \text{ and } f_i(x_1)=f_i(x_2)=y \text{ imply } \rho(y, M^r-\sigma_i)<\delta\,\frac{2k-i}{2k}\cdot$$

Suppose that $f_0, f_1, \cdots, f_{i-1}$ are already defined. Let

$$f_i(x)=f_{i-1}(x)\quad\text{if}\quad f_{i-1}(x)\in M^n-\sigma_i,$$

and let $Q_i^k=f_{i-1}^{-1}(\sigma_i)$.

M^r being a manifold, σ_i is simplicially homeomorphic with a convex r-cell in a euclidean r-dimensional space. Since $r>2n$, then using the very well known[6] procedure of making vertices linearly independent we find a simplicial map $f_i(Q_i^n)\subset\sigma_i$ such that $f_i(x)=f_{i-1}(x)$ if $f_{i-1}(x)$ is on the boundary of σ_i and satisfying (1)–(3).

Taking $g=f_k$ it follows from (1) that

$$|g(x)-f(x)|<\epsilon.$$

[4] With respect to certain simplicial subdivisions of M^r and S^n.

[5] σ_i consists of all closed simplices of M^r containing a_i.

[6] See for instance W. Hurewicz, Sitzungsberichte der Preussischen Akademie der Wissenschaften, vol. 24 (1933), p. 758.

Now if $x_1 \neq x_2$ and $g(x_1) = g(x_2)$, then according to (2) we have

$$f_i(x_1) = f_i(x_2) = y_i \qquad \text{for } i = 0, 1, \cdots, k.$$

Owing to the definition of δ there is an index $j = 0, 1, \cdots, k$ such that

$$\rho(y_0, M^n - \sigma_j) > \delta.$$

Combining this with (1) we see that

$$\rho(y_i, M^n - \sigma_j) > \delta - \frac{\epsilon i}{k} > \delta - \frac{\delta i}{2k} = \delta\, \frac{2k - i}{2k}\cdot$$

Taking $i = j$ we obtain a contradiction with (3).

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