ON A PROPERTY OF k CONSECUTIVE INTEGERS[1]

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S. S. Pillai[2] has just proved the following theorem: In every set of less than 17 consecutive integers there exists at least one integer which is relatively prime to all the others; there are sequences of k integers for $k=17, 18, \cdots, 430$, however, which have not this property. Pillai conjectures that the same is valid for every $k \geqq 17$. I shall prove that this conjecture is true.

The method of the proof is similar to the method I applied in a joint paper with H. Zeitz[3] in proving that the following conjecture is wrong for every prime $p \geqq 43$.

Denote by p_n the nth prime. Then there exist at most $2p_{n-1}-1$ consecutive integers such that each of these integers is at least divisible by one of the primes $p_1, p_2, \cdots, p_n$.

This conjecture was used by Legendre for his proof of the theorem of the primes in arithmetical progressions. First I prove the following.

LEMMA. *Let $\pi(x)$ be the number of primes $p \leqq x$. Then we have*

$$\pi(2x) - \pi(x) \geqq 2\left[\frac{\log x}{\log 2}\right] + 2 \tag{1}$$

for every $x \geqq 75$.

PROOF. If we put, as usual,

$$\vartheta(x) = \sum_{p \leqq x} \log p,$$

then we have

$$\begin{aligned} \pi(2x) - \pi(x) &= \sum_{x<p\leqq 2x} 1 \geqq \sum_{x<p\leqq 2x} (\log p/\log 2x) \\ &= \left\{ \sum_{x<p\leqq 2x} \log p \right\} /\log 2x = \left\{\vartheta(2x) - \vartheta(x)\right\}/\log 2x. \end{aligned} \tag{2}$$

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[1] Presented to the Society, September 12, 1940.

[2] S. S. Pillai, *On m consecutive integers*, Proceedings of the Indian Academy of Sciences, section A, vol. 11 (1940), pp. 6–12.

[3] A. Brauer und H. Zeitz, *Über eine zahlentheoretische Behauptung von Legendre*, Sitzungsberichte der Berliner mathematischen Gesellschaft, vol. 29 (1930), pp. 116–125. Cf. A. Brauer, *Question concerning the maximum term in the diatomic series—proposed by A. A. Bennett*, American Mathematical Monthly, vol. 40 (1933), pp. 409–410.

It is known that[4]

$$\vartheta(2x) - \vartheta(x) \geqq \vartheta(2x-2) - \vartheta(x) > .7x - 3.4x^{1/2} - 4.5\log^2 x - 24\log x - 32. \tag{3}$$

Hence, by (1), (2), and (3), it is sufficient to prove that

$$.7x - 3.4x^{1/2} - 4.5\log^2 x - 24\log x - 32 > (2\log x/\log 2 + 2)(\log x + \log 2),$$

$$f(x) = .7x - 3.4x^{1/2} - \log^2 x(4.5 + 2/\log 2) - 28\log x - 32 - 2\log 2 > 0.$$

It is easy to see that $f(x) > 0$ holds for $x = 1024$, since $\log 1024 < 7$. Moreover we have

$$f'(x) = .7 - 1.7x^{-1/2} - \frac{9 + 4/\log 2}{x}\log x - \frac{28}{x} > 0 \quad \text{for } x \geqq 1024.$$

Hence $f(x)$ is increasing for $x \geqq 1024$ and the lemma is proved for $x \geqq 1024$.

For $75 \leqq x < 1024$ the lemma can be proved directly. For instance, it follows for $591 \leqq x < 1024$ and for $355 \leqq x < 591$ by the fact that there are 22 primes between 1024 and 1182 and 20 primes between 591 and 710. In the same way we get the lemma for $231 \leqq x < 355$, $159 \leqq x < 231$, and so on.

THEOREM. *For every $k \geqq 17$ there exists a sequence of k consecutive integers such that none of these k integers is relatively prime to the product of the others.*

PROOF. In view of the paper of Pillai, it is sufficient to prove the theorem for $k \geqq 300$. We put

$$m = \left[\frac{k}{4}\right] \geqq 75. \tag{4}$$

Let $p_1, p_2, \cdots, p_r$ be the primes in the closed interval $\{1 \cdots m\}$ and $p_{r+1}, p_{r+2}, \cdots, p_s$ the primes in the closed interval $\{m+1 \cdots 2m\}$. If we consider k consecutive integers, then each of the primes

$$p_1, p_2, \cdots, p_r, p_{r+1}, p_{r+2}, \cdots, p_s \tag{5}$$

divides at least two of the k integers, since each of these primes is less than $2m$, hence by (4) less than $k/2$. Therefore each of these k integers which is divisible by at least one of the primes (5) is not relatively prime to all the $k-1$ other integers. Hence it is sufficient to

[4] See, for example, E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, vol. 1, 1909, p. 91.

prove that there exist sequences of k integers such that for $k \geqq 300$ each of these integers is divisible by at least one of the primes (5).

We consider the simultaneous congruences

$$x \equiv 1 \pmod{2}, \qquad x \equiv 0 \pmod{p_2 p_3 \cdots p_r}. \tag{6}$$

Let x be a solution of (6). Then the integers

$$x-2m,\ x-2m+2,\ \cdots,\ x-2,\ x,\ x+2,\ \cdots,\ x+2m-2,\ x+2m \tag{7}$$

form a sequence of $2m+1$ odd integers of the form

$$x \pm 2\mu, \qquad \mu = 0, 1, \cdots, m. \tag{8}$$

If μ is divisible by the odd prime p_ν, we have $p_\nu \leqq p_r$, since $\mu \leqq m$ because of (8). Hence we obtain from (6) that

$$x \pm 2\mu \equiv 0 \pmod{p_\nu}. \tag{9}$$

It follows from (9) that all those integers of (7) which have not the form $x \pm 2^\tau$ with $\tau \geqq 1$ are divisible by at least one of the primes $p_2, p_3, \cdots, p_r$.

If we put

$$\left[\frac{\log m}{\log 2}\right] + 1 = t, \tag{10}$$

then the integers of the form $x \pm 2^\tau$ with $\tau \geqq 1$ in the set (7) are the integers

$$x \pm 2,\ x \pm 2^2,\ \cdots,\ x \pm 2^t. \tag{11}$$

By (4), it follows from the lemma and from (10) that the number of primes in the closed interval $\{m+1 \cdots 2m\}$ is

$$\pi(2m) - \pi(m) \geqq 2\left[\frac{\log m}{\log 2}\right] + 2 = 2t.$$

On the other hand the primes in this interval were $p_{r+1}, p_{r+2}, \cdots, p_s$, hence

$$s - r \geqq 2t, \qquad p_{r+2t} \leqq p_s. \tag{12}$$

Beside the congruences (6) we now subject x to the following $2t$ congruences

$$\begin{aligned} x + 2^\tau &\equiv 0 \pmod{p_{r+\tau}}, \\ x - 2^\tau &\equiv 0 \pmod{p_{r+t+\tau}}, \end{aligned} \qquad \tau = 1, 2, \cdots, t. \tag{13}$$

These congruences always have solutions. For every solution x all the numbers (7) are divisible by at least one of the primes (5), since each of the integers (11) is divisible by at least one of the primes $p_{r+1}, p_{r+2}, \cdots, p_s$ because of (13) and (12).

Hence each of the $4m+3$ consecutive integers

$$x-2m-1,\ x-2m,\ x-2m+1,\ \cdots,\ x-1,\ x,\ x+1,\ \cdots,\ x+2m+1 \tag{14}$$

is divisible by at least one of the primes (5), since

$$\begin{aligned} x - 2m - 1 \equiv x - 2m + 1 \equiv \cdots \equiv x - 1 \equiv x + 1 \\ \equiv \cdots \equiv x + 2m + 1 \equiv 0 \pmod 2. \end{aligned}$$

Because of (4) we have

$$k \leqq 4m + 3.$$

Therefore we can take k consecutive integers from (14). None of these k integers is relatively prime to the product of the $k-1$ others.

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