ON THE ANALOGUE FOR DIFFERENTIAL EQUATIONS OF THE HILBERT-NETTO THEOREM

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If

$$(1) \qquad F_1, \cdots, F_r$$

is a finite system of differential polynomials in the unknowns $y_1, \cdots, y_n$, and if G is a differential polynomial which is annulled by every solution of the system (1), some power G^p of G is a linear combination of the F_i and their derivatives of various orders, with differential polynomials for coefficients. This analogue of the Hilbert-Netto theorem was proved by J. F. Ritt[1] for forms with meromorphic coefficients, and by H. W. Raudenbush[2] for the case of coefficients belonging to an abstract differential field. In these proofs it is shown that the denial of the existence of the exponent p, above, of G leads to a contradiction; no constructive method for obtaining admissible values of p is given. The object of the present note is to present a new proof of the analogue, for the case of meromorphic coefficients, which is entirely constructive and produces a definite G^p as described above.

Our proof will be based on the considerations in Chapters V and VII of A.D.E. In Chapter VII, the problem of obtaining G is reduced to the problem of determining unity as a linear combination of the F_i in (1) and their derivatives, in the case in which (1) has no solutions. In Chapter V it is shown how to decide, in a finite number of steps, whether or not (1) has solutions. Our problem thus assumes the following form: *Given that* (1) *has no solutions, it is required to express unity as a linear combination of the* F_i *and their derivatives.*

We assume that (1) has no solutions and proceed to examine the algorithm developed in §§65–67 of A.D.E. Adjoining to (1) a finite number of linear combinations of the F_i and their derivatives, we obtain a system Σ, devoid of solutions, with a basic set

$$(2) \qquad A_1, \cdots, A_q$$

which has the property that the remainder of every form in Σ with respect to (2) is zero. If (2) consists of a single form A which is an

[1] Ritt, J. F., *Differential Equations from the Algebraic Standpoint*, chap. 7, referred to below as A.D.E. American Mathematical Society Colloquium Publications, vol. 14, 1932.

[2] Raudenbush, H. W., *Ideal theory and algebraic differential equations*, Transactions of this Society, vol. 36 (1934), pp. 361–368.

element of $\mathcal{J}$, the coefficient field of our forms, we secure immediately a representation of the type desired for unity. Let us suppose that this is not the case. Then (2), considered as a set of simple forms, cannot be a basic set of a prime system; if it were, (1) would possess solutions (A.D.E., §65). Thus there must exist, for some $j \leqq q$, an identity

$$(3) \qquad J_1^{\mu_1} \cdots J_{j-1}^{\mu_{j-1}}(SA_j - H_1H_2) - L_1A_1 - \cdots - L_{j-1}A_{j-1} = 0,$$

where J_i is the initial of A_i, $i=1, \cdots, j-1$; and where H_1 and H_2 are reduced with respect to $A_1, \cdots, A_j$. Let $\Lambda_1^{(k)}$, $k=1, \cdots, j+1$, represent the systems $\Sigma + J_1, \cdots, \Sigma + J_{j-1}, \Sigma + H_1, \Sigma + H_2$, respectively. We treat each $\Lambda_1^{(k)}$ as (1) was treated. The adjunction of a finite number of forms to any $\Lambda_1^{(k)}$ produces a system $\Sigma_1^{(k)}$, with no solutions, and with basic sets lower than (2) which furnish zero remainders for the forms in $\Sigma_1^{(k)}$.

Let us suppose that each $\Sigma_1^{(k)}$ contains an element of $\mathcal{J}$ different from zero. We see on examining these systems that there exist relations, procurable by constructive methods,

$$(4) \qquad 1 = P + M_0H_1 + M_1H_1' + \cdots,$$

$$(5) \qquad 1 = Q + N_0H_2 + N_1H_2' + \cdots,$$

$$(6) \qquad 1 = R_i + S_{i0}J_i + S_{i1}J_i' + \cdots, \qquad i = 1, \cdots, j-1,$$

accents indicating differentiation, where P, Q, and the R_i are linear in the F_i and their derivatives.

We equate to unity the product of the right-hand members of (4), (5), and the equations (6). If both sides of the resulting equation are raised to a sufficiently high power, determinable in advance, we secure, as Raudenbush has shown, a relation of the type

$$1 = L + T_0V + T_1V' + \cdots,$$

accents indicating differentiation, where L is linear in the F_i and their derivatives, and $V = J_1^{\mu_1} \cdots J_{j-1}^{\mu_{j-1}}H_1H_2$.[3] From (3) we see that V can be obtained as a linear expression in the F_i and their derivatives. We thus have such an expression as we are seeking for unity.

If, on the other hand, some system $\Sigma_1^{(j)}$ does not contain a nonzero element in $\mathcal{J}$, we apply to it the entire process applied to Σ. We form in this way systems with basic sets lower than those of $\Sigma_1^{(j)}$. The systems thus formed for the various $\Sigma_1^{(k)}$ receiving our present treatment will be called, with no attempt to describe their complete history, systems Σ_2. In each Σ_2 a basic set yields only zero remainders.

[3] Ritt; J. F., *Algebraic aspects of the theory of differential equations*, Semicentennial Publications of the American Mathematical Society, vol. 2, p. 44.

Let us suppose that each Σ_2 contains a nonzero element in $\mathcal{J}$. What precedes shows that, for each $\Sigma_1^{(j)}$ as above, unity is linear in the forms of $\Sigma_1^{(j)}$ and their derivatives; this, again, gives the expression which we are seeking for unity.

If there are Σ_2 which contain no nonzero element in $\mathcal{J}$, we give them the treatment which is now familiar. By §67 of A.D.E., we know that our process can continue for only a finite number of steps, so that the possibility of determining for unity an expression of the type desired is established.

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