148. Stefan Bergman: *Numerical methods for conformal mapping of polygonal domains.*

The problem of the approximate determination of a conformal mapping can be reduced in certain cases to numerical operations which can be carried out with the aid of devices for computation already in existence or by electrical calculating machines combined, perhaps, with punch machines. In the present paper the calculation of the constants (that is, the branch points of the integrand) in the Schwarz-Christoffel formula (which transforms the half-plane into a polygonal domain) is reduced to the determination of the Fourier coefficients of $[R(\phi)]^n$, and to the solution of a system of linear equations. If the domain is a star domain, $R=R(\phi)$ is the equation of its boundary curve in polar coordinates. Further, methods for the calculation of the resulting integrals are discussed. The method is applied to a technical problem, namely to the problem of torsion in a beam with a polygonal section. The analogous method can be applied for the solution of boundary problems and has important applications in certain problems of aeronautics. (Received January 22, 1941.)

149. F. H. Clauser: *Exact solutions of the equations for the flow of a compressible fluid.* Preliminary report.

Several solutions of the equations given by Tschaplygin for the flow of a compressible fluid are discussed and a method presented for easily accomplishing the transformation of the solutions in the hodograph plane back to the physical plane. (Received December 18, 1940.)

Geometry

150. L. M. Blumenthal and G. E. Wahlin: *On the spherical surface of smallest radius enclosing a bounded subset of n-dimensional euclidean space.*

A short elementary proof is given for the theorem: *If M is any bounded subset of n-dimensional euclidean space E_n with positive diameter d, then there is a unique $(n-1)$-dimensional spherical surface of smallest radius r enclosing M, and $r \leqq [n/2(n+1)]^{1/2}\cdot d$.* In a proof abounding with algebraic difficulties, H. W. E. Jung established these results in 1901 for the case of finite point sets and indicated their extension to infinite sets at the end of his long paper (Journal für die reine und angewandte Mathematik, vol. 123 (1901), pp. 241–257). The simplification offered by the present proof is afforded in large measure by a lemma which shows that *if each $n+1$ points of a subset M in E_n may be enclosed by an $(n-1)$-dimensional spherical surface of radius r then M itself has this property.* The proof exhibits the geometrically simple nature of the theorem. (Received January 24, 1941.)

151. J. J. DeCicco: *Equilong geometry of differential equations of first order.*

With this paper the study of the equilong geometry of a field of lineal elements is begun. This may be considered to be an analogue of a preceding paper by Kasner and the author in which the conformal geometry of a field is developed. As defined by Kasner, a dual-isothermal family consists of ∞^1 curves which are equilongly equivalent to a pencil of circles (all those tangent to two fixed lines, distinct or coincident). Obviously all dual-isothermal families are equilongly equivalent. It is found that any non-dual-isothermal family possesses two absolute differential covariants of the third order. Conversely if two fields possess the same two covariants, then there exist $2\infty^2$ equilong correspondences which will carry one field into the other. For any two nearby nonparallel elements of the field $w=w(u, v)$, the general equilong distance is defined by $ds^2=w_{vv}du^2$, whereas for two nearby parallel elements, this distance is $ds^2=w_{vv}dv^2$. The final result is that in equilong geometry any absolute differential covariant of a field is a function of two fundamental covariants and their derivatives with respect to the equilong arc lengths of the unions and the equiparallel series of the field. (Received January 20, 1941.)

152. J. J. DeCicco: *Isodeviate systems of geodesic series.*

This paper is concerned with the derivation of some further results in the differential geometry of a field of lineal elements in the plane. A system of ∞^1 series of a given field, which corresponds by an equideviate transformation to a linear pencil of turbines of a flat field, is called an isodeviate system. If a field possesses at least one isodeviate system of geodesic series, then it must possess ∞^2 such systems. The condition necessary and sufficient for this is that the gaussian curvature K be the same at all parallel elements of a field. To compare this with the corresponding theory in the geometry of surfaces in euclidean three-space, see Kasner, *Isothermal systems of geodesics*, Transactions of this Society, vol. 5 (1904), pp. 56–60. (Received December 31, 1940.)

153. J. J. DeCicco: *Lineal element transformations which preserve dual-isothermal families.*

This paper is analogous to the one by Kasner and the author in which the group of all lineal element transformations which preserve isothermal families is determined. Kasner has defined a dual-isothermal family to consist of ∞^1 curves which are equilongly equivalent to a pencil of circles (in line geometry). In this work all lineal element transformations are found which preserve dual-isothermal families. In hessian or equilong coordinates this group is $U=\phi$, $V=(a_1v+b_1w+c_1)/(av+bw+c)$, $W=(a_2v+b_2w+c_2)/(av+bw+c)$, where ϕ, a, b, c, a_1, b_1, c_1, a_2, b_2, c_2 are functions of u only. The subgroup of contact transformations is $U=\phi(u)$, $V=v\psi(u)+\chi(u)$, obviously larger than the equilong group. Earlier the author had found that a non-dual-isothermal family possesses two absolute differential covariants of the third order. In the final part of the present paper, it is proved that any lineal element transformation which preserves both of these two covariants must be an equilong transformation. (Received January 21, 1941.)

154. J. J. DeCicco: *The differential geometry of the Laguerre group G_6.*

In the Laguerre inversive geometry of the plane the length of arc of a curve is defined in equilong coordinates (x, y) by $s=\int(y''')^{1/2}dx$. The Laguerre inversive curvature is $K=[4y'''y^{v}-5(y^{iv})^2]/4(y''')^2$. The fundamental result of the paper is that two curves are equivalent under the Laguerre group G_6 if and only if their curvatures are the same functions of the arc length. Thus $K=F(s)$ is the intrinsic equation of any curve. As the Laguerre group of the complex plane is isomorphic to the group of rigid motions of complex euclidean three-space with the curves of the plane corresponding to the minimal curves of spaces, the fundamental result also furnishes the intrinsic equations of the minimal curves of space. (Received December 31, 1940.)

155. J. M. Feld: *The geometry of whirls and whirl-motions in space.*

The geometry of the six-parameter whirl-motion group G_6 in the plane had its origin in Kasner's fundamental paper, *The geometry of turns and slides, and the geometry of turbines* (American Journal of Mathematics, vol. 33 (1911), p. 193). In a recent paper by Feld, *Whirl-similitudes, euclidean kinematics, and non-euclidean geometry* (this Bulletin, abstract 46-5-270), Kasner's G_6 was extended first to a mixed six-parameter group composed of eight distinct continuous families, one of which is G_6, and second to a seven-parameter mixed group of eight families—the *group of whirl-similitudes*. The latter group was shown to be isomorphic with the group of automorphisms of *pseudo-elliptic three-space*. Feld also presented a (1, 1) mapping of lineal elements, flat fields, and turbines on planar euclidean displacements, symmetries, and ordered pairs of points respectively. The subject of this paper is the development of an analogous theory for three-space, where a mixed eight-family twelve-parameter continuous group of whirl-motions plays the principal role. Quaternion geometry is employed. (Received January 14, 1941.)

156. Aaron Fialkow: *The foundations of the conformal differential geometry of a subspace.*

For an arbitrary subspace V_n in any Riemann space V_m $(0<n<m;\ m>2)$, the author shows that it is possible to define a system of differential forms, termed the *conformal fundamental forms* (c.f.f.'s) of V_n, which enjoy the following properties: (1) If $V_n \subset V_m$, $\overline{V}_n \subset \overline{V}_m$ and $V_m \leftrightarrow \overline{V}_m$, $V_n \leftrightarrow \overline{V}_n$ by a conformal map, then the c.f.f.'s of V_n and $\overline{V}_n$ are equal. (2) Conversely, if V_m and $\overline{V}_m$ are conformally euclidean spaces and the c.f.f.'s of V_n and $\overline{V}_n$ are equal, then a conformal transformation exists so that $V_m \leftrightarrow \overline{V}_m$, $V_n \leftrightarrow \overline{V}_n$. (3) A V_n exists with any preassigned c.f.f. whose coefficients satisfy certain conformally invariant partial differential equations analogous to the classical generalized Gauss-Codazzi equations. The results for $n \geqq 4$ are typical while the cases $n=3$, 2, and 1 respectively exhibit increasing degrees of deviation from the normal situation. These theorems serve as the basis for the development of a conformal differential geometry in many ways similar to classical differential geometry but also exhibiting a number of essential differences. The principal tool is a new simple type of differentiation (with respect to the subspace) which enjoys all the usual properties of covariant differentiation as well as a number of others which give it its distinctive conformal character. (Received January 25, 1941.)

157. M. C. Foster: *Note on autopolar surfaces.*

This paper is concerned with surfaces autopolar with respect to the paraboloid $2z=x^2+y^2$. Such autopolar surfaces are considered as special solutions of those partial differential equations which are invariant under the dual transformation for which the above paraboloid is the quadric of reference. Various metric properties are considered. (Received January 22, 1941.)

158. Edward Kasner and J. J. DeCicco: *Conformal geometry of differential equations of first order.*

In this paper the conformal geometry of nonisothermal fields of lineal elements in the plane is studied. A field F possesses two absolute conformal covariants of the third order. Conversely, if two fields F and G possess these same covariants at corresponding elements, then there exist $2\infty^2$ conformal transformations which will carry one into the other. These results may be interpreted geometrically as follows. If the equation of the field F is $\theta=\theta(x, y)$, where θ is the inclination, the conformal distance dS between two lineal elements of F is defined by $dS^2=\frac{1}{4}(\theta_{xx}+\theta_{yy})(dx^2+dy^2)$. With this definition of distance and the usual notion of angle, it results that the two covariants are the geodesic curvatures of the unions and the orthogonal series of F. All other covariants of F are functions of these two geodesic curvatures and their partial derivatives with respect to the conformal arc lengths of the unions and the orthogonal series. In the final part of the paper, many interesting conformal properties of n-webs of curves are obtained. (Received December 31, 1940.)

159. Edward Kasner and J. J. DeCicco: *Conformal geometry of velocity systems.*

In this paper the conformal geometry of velocity systems is studied. A set of ∞^2 curves is a velocity system if and only if the ∞^1 osculating circles of the ∞^1 curves of the set passing through a fixed point, constructed at this point, form a pencil. Any arbitrary point transformation for which neither of the two families of minimal lines is preserved carries exactly one velocity system into a velocity system. Any point transformation which preserves only one family of minimal lines converts ∞^1 velocity systems into velocity systems. Finally, the conformal group preserves all velocity systems. A velocity system may contain exactly ∞^2, ∞^1, one, or no isothermal systems. A velocity system is called a Γ family if it is conformally equivalent to the ∞^2 circles orthogonal to a fixed circle. A velocity system is a Γ family if and only if it possesses ∞^2 isothermal families. All the conformal covariants of a velocity system are obtained. Finally, the reciprocal system of any Γ system is treated. (Received December 31, 1940.)

160. Edward Kasner and J. J. DeCicco: *Infinite groups generated by equilong involutions and symmetries.*

In equilong geometry, the set of all equilong transformations of period two may be classified into three distinct types: equilong involutions, K symmetries, and D inversions. (Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 471–476.) In the present paper the infinite groups generated by these transformations are determined. The group K'_{sym} formed by all K symmetries consists of K symmetries and K translations (products of two K symmetries). Any transformation of the group G'_{invol} (D'_{invers}) formed by all equilong involutions (D inversions) can be factored into involutions (D inversions) in an infinitude of ways of which at least one can be factored into four or fewer involutions (D inversions). The fundamental result is that the group generated by all equilong transformations of period two is identical with the group generated by K symmetries and D inversions. To contrast this with the conformal theory, see Kasner, American Journal of Mathematics, vol. 38 (1916), pp. 177–184. (Received December 31, 1940.)

161. Edward Kasner and J. J. DeCicco: *Lineal element transformations which preserve isothermal families.*

In this paper the authors seek to generalize the well known result that the conformal transformations are the only *point correspondences* which carry every isothermal family into an isothermal family of curves. They find the group of all *lineal element transformations* of the plane (not necessarily of the contact type) which preserves all isothermal families. This group is given in (x, y, θ), where (x, y) are the cartesian coordinates of the point, and θ is the inclination of the element, by $X=\phi(x, y)$, $Y=\psi(x, y)$, $\Theta=k\theta+h(x, y)$, where k is a nonzero constant, ϕ and ψ satisfy the Cauchy-Riemann equations (direct or reverse), and h is any harmonic function. The only contact correspondences are the conformal ones. In the authors' previous work, it was shown that any non-isothermal field possesses two absolute differential covariants of the third order under the conformal group. In the final part of the present paper it is proved that a lineal element transformation which preserves either one of these two covariants must be a conformal transformation. (Received January 20, 1941.)

162. Edward Kasner and J. J. DeCicco: *The classification of analytic arcs or elements under the group of arbitrary point transformations.*

This paper begins the study of the invariant theory of a single irregular analytic arc or element based on the infinite group of arbitrary point transformations in the plane. The correspondences are (regular) analytic in x and y. Then y may be written as a power series which proceeds according to positive integral powers of the pth root of x. Let q be the first power of the pth root of x which is not a multiple of p. The index p and the rank q are arithmetic invariants; and all elements with the same p and q form the single species (p, q). Absolute differential invariants exist for all irregular species except in the cases $(4, 5)$, $(4, 6)$, $(4, 7)$, $(3, q)$, and $(2, q)$. The species $(4, 5)$ and $(4, 7)$ may be separated into two and three distinct sets respectively. The species $(4, 6)$ and $(3, q)$ possess an additional arithmetic invariant. All the elements of the species $(2, q)$ are equivalent. The paper will appear in the Proceedings of the National Academy of Sciences. (Received December 12, 1940.)

163. Don Mittleman: *Theory of ortho-family: A generalization of natural family.*

A surface will be said to be mapped orthogonally onto a plane if the image of an orthogonal net on the surface is the rectangular cartesian net of the plane. An ortho-family is defined as the image of the geodesics of the surface under the particular orthogonal mapping. The necessary and sufficient conditions that a two-parameter family of curves in the plane be an ortho-family are given analytically. It is a simple consequence of these conditions that an ortho-family which is a velocity family must be a natural family. Finally, if an ortho-family which is obtained by an orthogonal, non-conformal mapping is a natural family, then the surface whose geodesics are the pre-image of the given natural family is a surface of Liouville referred to the coordinate system for which $ds^2=(1/f-1/g)(dx^2/f+dy^2/g)$, where f is a function of x alone, and g a function of y alone. (Received January 24, 1941.)

164. Nelson Robinson: *On the contact of a quartic surface with a general analytic surface.*

Using the differential equations of a general analytic surface S referred to its asymptotic net, the power series development of S is computed to terms of the sixth degree. By means of this expansion conditions are determined in order that a general quartic surface have various orders of contact with the original surface S at a point P. Special cases of the quartics Q are discussed with particular attention given to the cases where Q is composite, the components being quadric surfaces. Necessary and sufficient conditions are found in order that Q be composed of quadrics of Darboux, or quadrics of Lie. (Received December 30, 1940.)

165. Peter Scherk: *On real closed curves of order $n+1$ in projective n-space.* II. Preliminary report.

In the first part of this paper (abstract 46-11-502) the author discussed differentiable closed curves K^{n+1} of real order $n+1$ in R_n by means of a certain single-valued correspondence of the K^{n+1}. He proved that $S \leqq n+1$, $S \equiv n+1 \pmod 2$ if S is the sum of the multiplicities of the singular points, and he characterized the case $S=n+1$. Extending a simple remark on rotation numbers to multi-valued correspondences, the author discusses a two-valued and a three-valued correspondence defined on certain arcs of the K^{n+1} and on the whole K^{n+1} respectively, and connected with the projections of the K^{n+1} from its osculating $(n-2)$-spaces and $(n-3)$-spaces respectively. The study of these two correspondences yields: (1) the first estimates of the number of osculating $(n-2)$-spaces which meet the K^{n+1} again; (2) the classification of the K^{n+1} with $S=n-1$; (3) the classification of the K^6; (4) a more systematic access to the classification of the K^4 (previously obtained by the author). (Received January 24, 1941.)

166. Alexander Wundheiler: *Abstract algebraic definition of an affine vector space.* Preliminary report.

A linear set over the field of real numbers will be called a simple vector space, and its elements, simple vectors. Two simple vector spaces A and B are cogrediently coupled if for any a in A and b in B a real number $f(a, b)$ is defined, such that $f(ka, b)=f(a, kb)=kf(a, b)$; $f(a, b'+b'')=f(a, b')+f(a, b'')$; $f(a'+a'', b)=f(a', b)+f(a'', b)$. The a's and b's are then contragredient vectors. If A and B are of the same dimension, the set $A+B$ is called an affine vector space, a is a contravariant affine vector, b a covariant one, or vice versa. Various illustrations are given, as electrical networks, the space of fruit juice cocktail cans, and so on. (Received January 24, 1941.)

167. Oscar Zariski: *Pencils on an algebraic variety and a new proof of a theorem of Bertini.*

The theorem of Bertini-Enriques states that if a linear system of W_{r-1}'s on a V_r is reducible (that is, every W_{r-1} of the system is reducible) and is free from fixed components, then the system is composite with a pencil. In this paper a new proof of this theorem is given, together with an extension to irrational pencils. With every pencil $\{W\}$ there is associated a field P of algebraic functions of one variable, a subfield of the field Σ of rational functions on V_r. The essential point of the proof is the remark that $\{W\}$ is composite if and only if P is not maximally algebraic in Σ. The rest of the proof, in the case of pencils, follows from the fact that an irreducible algebraic variety V_r over a ground field K is absolutely irreducible if K is maximally algebraic in Σ. In the case of linear systems of dimension >1, the proof is based on the following lemma: if K is maximally algebraic in Σ and if x_1, x_2 are algebraically independent elements of Σ, then for all but a finite number of elements c in K the field $K(x_1+cx_2)$ is maximally algebraic in Σ. (Received December 12, 1940.)

Logic and Foundations

168. Alvin Sugar: *Postulates for the calculus of binary relations in terms of a single operation.*

In a recent paper (*Postulates for the calculus of binary relations*, Journal of Symbolic Logic, vol. 5 (1940), pp. 85–97) J. C. C. McKinsey gave a set of postulates for the