The max $|l_1^{(n)}(x)|$ is attained at $x=\pm 1$ since[4] (I) $\theta_{k+1}-\theta_k \leqq 2\pi/(2n+\alpha+\beta-1)$ provided $\frac{1}{2}\leqq\alpha$, $\beta\leqq\frac{3}{2}$ and $x_k\equiv\cos\theta_k$. Using the second asymptotic formula and the fact[4] that $n\theta_k\rightarrow j_k$ as $n\rightarrow\infty$ where j_k is the kth positive zero of $J_{\beta-1}(x)$, we find that

$$|l_k^{(n)}(1)| \rightarrow (\tfrac{1}{2}j_k)^{\beta-2}\,|\Gamma(\beta)J_\beta(j_k)|^{-1} \qquad \text{as } n\rightarrow\infty,\ k \text{ constant},$$

$l_1^{(n)}(-1)\rightarrow 0$ which proves the theorem:

THEOREM 7. *Max* $|l_1^{(n)}(x)|\rightarrow(\frac{1}{2}j_1)^{\beta-2}|\Gamma(\beta)J_\beta(j_1)|^{-1}$ *as* $n\rightarrow\infty$ *(where* $\frac{1}{2}\leqq\alpha$, $\beta\leqq\frac{3}{2}$, j_1 *is first positive zero of* $J_{\beta-1}(x)$*).*

A similar result holds for $l_n^{(n)}(x)$ if β is replaced by α.

For Legendre polynomials $(\alpha=\beta=1)$ this limit is approximately 1.602. For $\alpha=\beta=\frac{1}{2}$ and $\alpha=\beta=\frac{3}{2}$ the limit of Theorem 7 is also an upper bound for max $|l_1^{(n)}(x)|$ and max $|l_k^{(n)}(x)|$. Whether this is true, in general, remains unanswered.

PURDUE UNIVERSITY

AN INVARIANCE THEOREM FOR SUBSETS OF S^n [1]

SAMUEL EILENBERG

The purpose of this paper is to establish the following.

INVARIANCE THEOREM. *Let A and B be two homeomorphic subsets of the n-sphere S^n. If the number of components of S^n-A is finite, then it is equal to the number of components of S^n-B.*

In the case when A and B are closed this theorem is a very well known consequence of Alexander's duality theorem and its generalizations. In our case we also derive our result as a consequence of a duality theorem. However, the duality is established only for the dimension $n-1$.

Given a metric space X we shall say that Γ^k is a k-cycle in X if there is a compact subset A of X such that Γ^k is a k-dimensional convergent (Vietoris) cycle in A with coefficients modulo 2. We shall write $\Gamma^k\sim 0$ if $\Gamma^k\sim 0$ holds in some compact subset of X. The homology group of X obtained this way will be denoted by $\mathfrak{H}^k(X)$; the corresponding connectivity number, by $p^k(X)$. The number $p^k(X)$ can be either finite or ∞.

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[1] Presented to the Society, December 28, 1939.

DUALITY THEOREM. *Let* $A \subset S^n$ *and let* $z_0, z_1, \cdots, z_m$ *belong to* $m+1$ *different quasi-components*[2] *of* $S^n - A$. *There are* m *linearly independent* (*modulo* 2) $(n-1)$-*cycles*

$$(1) \qquad \Gamma_1^{n-1}, \cdots, \Gamma_m^{n-1}$$

of A *such that*

$$(2) \qquad v(\Gamma_i^{n-1}, \overset{0}{\gamma_j}) = \delta_{ij}, \qquad i, j = 1, \cdots, m,$$

where γ_j^0 *is the* 0-*cycle* $z_0 + z_j$ (*consisting of the two points* z_0, z_j *each of them with coefficient* 1) *and* $v(\Gamma, \gamma)$ *is the linking number.*

In case $S^n - A$ *has only* $m+1$ *quasi-components, the cycles* (1) *form a basis for* $\mathfrak{H}^{n-1}(A)$.

PROOF. In case A is closed the theorem turns out to be a particular case of the generalized Alexander duality theorem.[3] We shall prove our theorem for arbitrary sets A using the theorem for closed sets.

Since $z_0, z_1, \cdots, z_m$ belong to $m+1$ different quasi-components of $S^n - A$ there is a decomposition $S^n - A = A_0 + A_1 + \cdots + A_m$ such that $z_i \in A_i$ and $A_i\bar{A}_j + \bar{A}_iA_j = 0$ for $i \neq j$, $i, j = 0, 1, \cdots, m$. Let $B_0, B_1, \cdots, B_m$ be open disjoint sets such that $A_i \subset B_i$ for $i = 0, 1, \cdots, m$ and let $B = S^n - (B_0 + B_1 + \cdots + B_m)$. Clearly B is a closed subset of A and $z_0, z_1, \cdots, z_m$ belong to $m+1$ different quasi-components (equals components) of $S^n - B$.

Applying the duality theorem to the closed set B we obtain the cycles (1) satisfying (2). In order to prove that they determine linearly independent elements modulo 2 of $\mathfrak{H}^{n-1}(A)$ consider a cycle $\Gamma^{n-1} = a_1\Gamma_1^{n-1} + \cdots + a_m\Gamma_m^{n-1}$ where $a_j = 0, 1$. It follows from (2) that $v(\Gamma^{n-1}, \gamma_j^0) = a_j$. Therefore $\Gamma^{n-1} \sim 0$ in A implies $a_1 = \cdots = a_m = 0$.

Suppose now that $S^n - A$ consists of exactly $m+1$ quasi-components. It follows that the sets $A_0, A_1, \cdots, A_m$ are connected.

Let Γ^{n-1} be an $(n-1)$-cycle of A contained in some closed set $D \subset A$. Let E_i be the component of $S^n - (B+D)$ containing A_i $(i = 0, 1, \cdots, m)$ and let $E = S^n - (E_0 + E_1 + \cdots + E_m)$. It follows that (1°) E is a closed subset of A, (2°) $S^n - E$ consists of exactly $m+1$ quasi-components (equals components), (3°) the points

[2] Two points $x_1, x_2 \in X$ belong to the same quasi-component of X if there is no decomposition $X = A_1 + A_2$ such that $x_1 \in A_1$, $x_2 \in A_2$ and $A_1\bar{A}_2 + \bar{A}_1A_2 = 0$. If the number of quasi-components of X is finite then every quasi-component is a component.

[3] Alexander, J. W., Transactions of this Society, vol. 23 (1922), pp. 333–349; Frankl, F., Sitzungsberichte der Wiener Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse, 2A, vol. 136 (1927), pp. 689–699; Alexandroff, P., Annals of Mathematics, (2), vol. 30 (1928), p. 163.

$z_0, z_1, \cdots, z_m$ belong to different quasi-components of $S^n - E$, (4°) the cycles (1) and Γ^{n-1} are contained in E. According to the duality theorem for closed sets the cycles (1) form a basis for $\mathcal{H}^{n-1}(E)$. This implies the existence of $a_1, a_2, \cdots, a_m$ ($a_i = 0, 1$) such that

$$\Gamma^{n-1} \sim a_1\Gamma_1^{n-1} + \cdots + a_m\Gamma_m^{n-1} \text{ in } E.$$

This proves the theorem since $E \subset A$.

Given a metric space X let the number $b_0(X)$ be defined as follows:

$$\begin{aligned} b_0(X) &= 0 \quad \text{if } X = 0, \\ b_0(X) &= m \quad \text{if } X \neq 0 \text{ and } X \text{ has exactly } m+1 \text{ components}, \\ b_0(X) &= \infty \quad \text{if } X \text{ has an infinity of components.} \end{aligned}$$

Clearly the value of $b_0(X)$ remains unchanged if we replace in its definition components by quasi-components. The duality theorem implies therefore the following:

(I) *For every subset A of S^n we have*

$$p^{n-1}(A) = b_0(S^n - A).$$

(II) *For every two homeomorphic subsets A and B of S^n we have*

$$b_0(S^n - A) = b_0(S^n - B).$$

The invariance theorem stated in the introduction follows directly from (II).

If X consists of an infinity of components, then instead of taking $b_0(X) = \infty$ we could define $b_0(X)$ to be the cardinal number corresponding to the class of all components of X. Similarly $p^k(X)$ could be redefined as a cardinal number. But with these new definitions (I) and (II) are no longer true.[4] In fact, let A be a subset of S^1 such that $S^1 - A$ is closed and enumerably infinite, and let B be a subset of S^1 such that $S^1 - B$ is perfect and non-dense. It is clear that A and B are homeomorphic, that $b_0(S^1 - A) = p^0(A) = p^0(B) = \aleph_0$, and that $b_0(S^1 - B) = 2^{\aleph_0}$.

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[4] That (II) is no longer true was first pointed out to me by Dr. L. Zippin.