CONTINUA OF FINITE DEGREE AND CERTAIN PRODUCT SETS[1]

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The continua of finite degree have been studied and identified with certain well known classes of continua in a paper by G. T. Whyburn.[3] The author has shown that the continua of finite degree are identical with the continua homeomorphic to a continuum of finite length.[4] The object of the present note is to obtain other internal characterizations of these continua.

The symbol M will represent a (compact) continuum. The continuum M is said to be of finite degree at the point $p \in M$ provided that to each $\epsilon > 0$ there corresponds an *uncountable* family of neighborhoods (U) of p such that (a) the diameter of each neighborhood is less than ϵ, (b) each $F(U)$ is finite, where $F(U)$ is the boundary of U, and (c) for any pair of neighborhoods U and V either $\overline{U} \subset V$ or $\overline{V} \subset U$. If every point is of finite degree, the continuum M is said to be of finite degree. The characterization which we find most useful below is that a continuum M is of finite degree if and only if every subcontinuum contains uncountably many local separating points of M.

It will be shown that the classes of continua defined by each of the following properties are identical with the continua of finite degree.

PROPERTY N^0. *M is locally connected and to each pair of closed, disjoint subsets A and B in M there corresponds a finite collection of disjoint, perfect sets $H^1, H^2, \cdots, H^k$ such that any continuum K in M intersecting both A and B contains some H^i.*

PROPERTY Q. *If K and K_i, $(i = 1, 2, \cdots)$, are nondegenerate continua in M with $\lim K_i = K$, there exists an integer n such that $\prod_n^\infty K_i$ is an uncountable set.*

It will be noted that the Property N^0 is highly analogous to Property N which characterizes the locally connected continua such that no true cyclic element has a continuum of condensation.[5]

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[3] G. T. Whyburn, *Concerning continua of finite degree and local separating points*, American Journal of Mathematics, vol. 57, pp. 11–16.

[4] See abstract 45-9-321, this Bulletin.

[5] This concept is due to R. L. Moore. See his *Fundamental Point Set Theorems*, Rice Institute Pamphlets, vol. 23, no. 1, 1936.

For this characterization see G. T. Whyburn, *On continua of condensation*, American Journal of Mathematics, vol. 58, pp. 705–708.

THEOREM A. *A continuum is of finite degree if and only if it has Property* N^0.

PROOF. A continuum of finite degree has the property that any pair of closed, disjoint subsets A and B in M can be separated by a finite number of points $q^1, q^2, \cdots, q^n$ which are local separating points of degree two.[3] Suppose M is of finite degree. Set $d=\min \rho(q^i, q^j)$, $i\neq j$, $\rho(q^i, A+B)$. Since q^i is of degree two, there exists an uncountable family of neighborhoods $[U(q^i)]$ such that (a) $\delta(U)<\frac{1}{2}d$, (b) $F(U)$ consists of at most two points, and (c) for any pair of distinct elements U and V of the family either $\overline{U}\subset V$ or $\overline{V}\subset U$. Set

$$Y^i = \overline{\sum F(U(q^i))},$$

i fixed. Let H^i be a perfect subset of Y^i. There is thus determined a finite collection of disjoint, perfect sets $H^1, H^2, \cdots, H^n$. Let K be any continuum in M intersecting both A and B. Since M is of finite degree (thus hereditarily locally connected), K may be taken as an arc. Since K contains some q^i, it contains the corresponding H^i.

Conversely, suppose M has Property N^0. To prove M is of finite degree it suffices to show that every nondegenerate subcontinuum in M contains uncountably many local separating points of M. Let K be a continuum in M containing the distinct points x and y. Set $A=x$, $B=y$. Then by Property N^0 there exists disjoint, perfect sets $H^1, H^2, \cdots, H^n$ such that $K\supset H^1$ (say). If $K\cdot L$ is uncountable, where L is the set of local separating points of M, our end is attained. If $K\cdot L$ is countable, there is a point $z^1\in H^1(M-L)$. Set $d=1/2 \min \rho(x, H^i), \rho(y, H^i), \rho(H^i, H^j), i\neq j$. Let V be a region in M containing z^1 which is of diameter less than d. Set $d^1=\rho[z^1, F(V)]$. Let W be the $\frac{1}{2}d^1$ neighborhood of z^1, that is, the set of all points b such that $\rho(z^1, b)<\frac{1}{2}d^1$. Let X^1 and Y^1 be the components of $K-KW$ containing x and y, respectively. Since X^1 and Y^1 each contain a point of V and $z^1\in M-L$, there exists an arc s^1 in $V-z^1$ joining X^1 and Y^1. Hence $K^1=X^1+s^1+Y^1$ is a continuum from x to y not containing H^1. If K^1 contains no other H^i, we have a contradiction to Property N^0. If $K^1\supset H^2$, then $H^2\subset X^1+Y^1$ on account of the manner of selection of s^1. Since $K\cdot L$ is countable, $(X^1+Y^1)L$ is countable, hence there exists a point $z^2\in H^2(M-L)$. Using the same d as before, a continuum K^2 is constructed (with K^1 replacing K) such that K^2 contains neither H^1 nor H^2 and K^2 intersects both A and B. Performing these steps (with z^2 replacing z^1) will clearly give a continuum $K^2=X^2+s^2+Y^2$ not containing H^2, and since $s^2H^1=0$ and $X^2+Y^2\subset K^1$, we have K^2 contains neither H^1 nor H^2. After a finite number of such steps a continuum K^j is obtained which intersects both A and B and contains no H^i. This completes the proof that having Property N^0 is equivalent to being of finite degree.

THEOREM B. *A continuum is of finite degree if and only if it has Property* Q.

PROOF. Let M be of finite degree. Since this implies that M is hereditarily locally connected, it may be assumed without loss of generality that the continua K_i such that $\lim K_i = K$ are arcs. Let the end-points of K_i be a_i and b_i. Suppose $\lim a_i = x$ and $\lim b_i = y$. Since K by assumption is a nondegenerate continuum, we may take $x \neq y$. In a continuum of finite degree the points of degree two are dense on every subcontinuum, hence let z be a point of degree two in $K-(x+y)$. Let X, Y and Z be neighborhoods of x, y and z respectively, such that $(\overline{X}\cdot\overline{Z}+\overline{X}\cdot\overline{Y}+\overline{Y}\cdot\overline{Z})=0$. To Z there corresponds an uncountable family of neighborhoods (U) such that $\overline{U}\subset Z$ and $F(U)$ consists of at most two points. There is a definite $V=V(z)$ such that $\overline{V}\subset U$ for uncountably many U. Since $z\in\lim K_i$, there is an integer n such that arcs K_i, $i\geqq n$, intersect V, hence if we take $m\geqq n$ so large that $a_i\in X$, $b_i\in Y$ for $i\geqq m$, then K_i must contain two boundary points of each U for which $V\subset U\subset Z$. Thus $\prod_m^\infty K_i$ is uncountable.

To show that Property Q implies finite degree we show that infinite degree implies Property non-Q. It is clear that a continuum containing a convergence continuum has Property non-Q, hence we need consider only hereditarily locally connected continua. Let T be an arc in the continuum M which contains only a countable number of the local separating points of M. For each positive integer n let W_n be the set of all points of M at a distance not greater than $1/n$ from T. Let M_n be the component of W_n which contains T. Let L_n be the set of all points which separate the end-points a and b of T in M_n. Then in M_n there exist arcs s_n and t_n from a to b such that[6] $s_n t_n = a + L_n + b$. Set $K_{2n-1}=s_n$ and $K_{2n}=t_n$. Then $\lim K_n = T$. But $\prod_n^\infty K_i$ is at most countable, no matter what positive integer n is. For K_{2n-1} and K_{2n} have only a countable number of common points, namely, $a+L_n+b$. Hence M does not have Property Q.

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[6] G. T. Whyburn, *Some properties of continuous curves*, this Bulletin, vol. 33 (1927), pp. 305–308. See Theorem 3.