A CORRECTION TO "A NOTE ON LINEAR FUNCTIONALS"[1]

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R. S. Phillips has called our attention to an error in our paper *A note on linear functionals*. On page 526, we have misquoted a theorem of Lebesgue's: the statement in the last display on that page is incorrect. It is, in fact, contradicted by the Riemann-Lebesgue theorem whenever the functions $x_n(t)$ are the elements of a uniformly bounded orthonormal set. Fortunately, however, the error does not affect the validity of any of our results. The correct consequence of Lebesgue's theorem is that

$$\sup_{0 \leqq n < \infty} \ \sup_{0 \leqq t \leqq 1}{}^{0} \ | x_n(t) | < \infty; \tag{1}$$

that is, that $\sup_{0 \leqq n < \infty} \|x_n\|_B < \infty$. From this it still follows that any linear functional on B is a linear functional on R; and we used our incorrect statement only to deduce this. This consequence is true in virtue of the following simple lemma.

LEMMA. *If a set $\{x\}$ forms a normed vector space under two norms, $\|x\|$ and $\|x\|_B$, and if $\lim_{n\to\infty}\|x_n\| = 0$ implies that $\sup_{0 \leqq n<\infty}\|x_n\|_B < \infty$, then any distributive functional continuous with respect to the second norm is also continuous with respect to the first norm.*

PROOF. Let f be a distributive functional, continuous with respect to the norm $\| \cdots \|_B$, so that for some number H,

$$| f(x) | \leqq H\|x\|_B \tag{2}$$

for every x. Suppose that f is not continuous with respect to the norm $\| \cdots \|$; then, as is well known (cf. S. Banach, *Théorie des Opérations Linéaires*, 1932, p. 55) there exist elements y_n such that $\|y_n\| = 1$, $|f(y_n)| > n$. The elements $z_n = n^{-1/2}y_n$ have the properties

$$\|z_n\| \to 0, \tag{3}$$

$$| f(z_n) | > n^{1/2}. \tag{4}$$

By hypothesis, (3) implies that $\|z_n\|_B < K$, $n = 0, 1, 2, \cdots$, for some finite K. Then, by (2), $|f(z_n)| \leqq HK$, contradicting (4) for large n.

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[1] This Bulletin, vol. 44 (1938), pp. 523–528.