ON THE CONVERSE OF THE TRANSITIVITY OF MODULARITY

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E. H. Moore's theorem on the transitivity of modularity is as follows: Consider the basis[1] $\mathfrak{A}$, $\mathfrak{P}$, ϵ; if a positive hermitian matrix ϵ_0 is modular as to $\epsilon\ \epsilon$, then every vector which is modular as to ϵ_0 is modular[2] as to ϵ (that is, $\mathfrak{M}_{\epsilon_0} \subset \mathfrak{M}_{\epsilon}$).

In his doctoral thesis, the author establishes the converse of the preceding theorem as a consequence of the Hellinger-Toeplitz theorem.[3] In this note, we give a new proof for the converse of the transitivity of modularity, and then deduce the generalized Hellinger-Toeplitz theorem as a corollary. The converse of the transitivity of modularity is, therefore, equivalent to the Hellinger-Toeplitz theorem. We also establish the converse of the transitivity of modularity for matrices, and a theorem on the transitivity of accordance and finiteness.

THEOREM I. *Consider the basis* $\mathfrak{A}$, $\mathfrak{P}$, ϵ; *and let* ϵ_0 *be a positive hermitian matrix. Then the following assertions are equivalent*:

(1) *every vector* μ_0 *modular as to* ϵ_0 *is modular as to* ϵ;

(2) ϵ_0 *is modular as to* $\epsilon\ \epsilon_0$;

(3) ϵ_0 *is modular as to* $\epsilon\ \epsilon$.

If one of the preceding conditions is satisfied, the modulus of ϵ_0 *as to* $\epsilon\ \epsilon$ *is equal to the norm of* ϵ_0 *as to* $\epsilon\ \epsilon_0$.

In the course of demonstration, we let $\mathfrak{M}_0$ denote the space of vectors μ_0 modular as to ϵ_0; J_0, the integration process based on ϵ_0; and M_0, the modulus as to ϵ_0. Similar interpretations are given to the symbols $\mathfrak{M}$, J, M, for the base matrix ϵ. A vector which is finite as to ϵ is denoted by β.

If every μ_0 is modular as to ϵ, the matrix ϵ_0 is of type $\mathfrak{M}_0\overline{\mathfrak{M}}$. Then $J\epsilon_0\beta$ is in $\mathfrak{M}_0$ for every β, and $J_0(J\bar{\beta}\epsilon_0)\mu_0 = J\bar{\beta}J_0\epsilon_0\mu_0 = J\bar{\beta}\mu_0$ for every pair β, μ_0. Consequently, for every β, $M_0J\epsilon_0\beta$ is equal to the least upper bound of $|J\bar{\beta}\mu_0|$ for all μ_0 such that $M_0\mu_0 \leqq 1$, by part (2) of Theorem (41.9) in G.A. Similarly, for every μ_0, which is modular as to ϵ by hypothesis, $M\mu_0$ is equal to the least upper bound of $|J\bar{\beta}\mu_0|$

[1] E. H. Moore, *General Analysis* (G.A. for abbreviation), Part I, p. 4, and Part II, p. 84.

[2] Theorem (46.4), part (1) in G.A., II, p. 137.

[3] *Spaces associated with non-modular matrices with applications to reciprocals*, Chicago thesis, 1931, pp. 3–9. The same proof is given in G.A., II, p. 193.

for all β such that $M\beta \leqq 1$. If the class $\mathfrak{L}$ is identified as the class of vectors $\bar{\beta}$ such that $M\beta \leqq 1$, and F_l, on $\mathfrak{M}_0$ to $\mathfrak{N}$, is defined to be $(|J\bar{\beta}\mu_0|\,|\mu_0)$ for every $l=\bar{\beta}$, then by Theorem (53.55) in G.A.,[4] the upper bound of $M_0 J\epsilon_0\beta$, for all β finite as to ϵ such that $M\beta \leqq 1$, is finite. By Theorem (46.85) in G.A., ϵ_0 is modular as to $\epsilon_0\ \epsilon$. Since ϵ_0 is hermitian, ϵ_0 is also modular[5] as to $\epsilon\ \epsilon_0$.

When condition (2) is true, then condition (3) is secured by a simple application[6] of the composition of modularity[7] to $\epsilon_0 = J_0\epsilon_0\epsilon_0$. That the last condition implies the first is proved in Theorem (46.4) of Moore's G.A.

From $\epsilon_0 = J_0\epsilon_0\epsilon_0$ and part (2) of Theorem (46.9) in G.A., we have $N_{\epsilon_0\epsilon}\epsilon_0 = M_{\epsilon\epsilon}J_0\epsilon_0\epsilon_0 = M_{\epsilon\epsilon}\epsilon_0$. This completes the proof.

The hypothesis of the preceding theorem is assumed for the following corollary:

COROLLARY. *Let $\mathfrak{M}_{0*}$ consist of all μ_0 whose moduli as to ϵ_0 are bounded by a fixed constant. If $\mathfrak{M}_{0*}$ is a subset of $\mathfrak{M}$, then the moduli as to ϵ of all vectors in $\mathfrak{M}_{0*}$ are also bounded.*

We may assume, without losing generality, that the moduli of all vectors in $\mathfrak{M}_{0*}$ are at most unity. Since the spaces $\mathfrak{M}_0$ and $\mathfrak{M}$ are linear, the condition that every μ_0 for which $M_0\mu_0 \leqq 1$ is modular as to ϵ is equivalent to condition (1) in the preceding theorem. Consequently ϵ_0 is modular as to $\epsilon\ \epsilon_0$. The equation $\mu_0 = J_0\epsilon_0\mu_0$ gives, by Theorem (46.7) in G.A., that $M\mu_0 \leqq M_{\epsilon\epsilon_0}\epsilon_0$ whenever $M_0\mu_0 \leqq 1$.

THEOREM II. *Consider the basis $\mathfrak{A}$, $\mathfrak{P}^1$, $\mathfrak{P}^2$, ϵ^1, ϵ^2; and let ϵ_0^1, ϵ_0^2 be two positive hermitian matrices. Then the following assertions are equivalent*:

(1) *every matrix κ^{12} modular as to $\epsilon_0^1\ \epsilon_0^2$ is of type $\mathfrak{M}^1\overline{\mathfrak{M}}^2$*;

(2) *ϵ_0^1 is modular as to $\epsilon^1\ \epsilon^1$, and ϵ_0^2 is modular as to $\epsilon^2\ \epsilon^2$*;

(3) *every matrix κ^{12} modular as to $\epsilon_0^1\ \epsilon_0^2$ is modular as to $\epsilon^1\ \epsilon^2$.*

For the demonstration of the theorem, we shall show that $(1)\rightarrow(2)\rightarrow(3)\rightarrow(1)$. The second implication is proved in part (2) of Theorem (46.4) in G.A. The last implication follows from the fact that every matrix κ^{12} modular as to $\epsilon^1\ \epsilon^2$ is of type $\mathfrak{M}^1\overline{\mathfrak{M}}^2$. To show

[4] See also Hildebrandt, *On uniform limitedness of sets of functional operations*, this Bulletin, vol. 29 (1923), pp. 309–315; Fréchet, *Sur les fonctionelles bilinéaires*, Transactions of this Society, vol. 16 (1915), pp. 217–218.

[5] By a similar reasoning, we may, of course, deduce the Hellinger-Toeplitz theorem as a consequence of Theorem (53.55).

[6] See the author's thesis, loc. cit., p. 8, or Moore, G.A., II, p. 193.

[7] Moore, G.A., II, p. 144.

the first implication, consider any $\mu_0^1, \mu_0^2 \neq 0^2$ which are modular as to $\epsilon_0^1, \epsilon_0^2$ respectively, Theorem (47.2) in G.A. shows that $\mu_0^1\bar{\mu}_0^2$ is modular as to $\epsilon^1\ \epsilon^2$, and hence by hypothesis, $\mu_0^1\bar{\mu}_0^2$ is of type $\mathfrak{M}^1\overline{\mathfrak{M}}^2$. Since $\mu^2 \neq 0^2$, let $a \equiv \mu^2(p^2) \neq 0$. Then $\mu_0^1 \cdot a$, and hence μ_0^1, is modular as to ϵ^1. This proves that every μ_0^1 modular as to ϵ_0^1 is modular as to ϵ^1. By Theorem I, ϵ_0^1 is modular as to $\epsilon^1\ \epsilon^1$. Similarly, we prove that ϵ_0^2 is modular as to $\epsilon^2\ \epsilon^2$. The proof was suggested by Dr. Coral.

THEOREM III. (Generalized Hellinger-Toeplitz theorem.) *Consider the basis* $\mathfrak{A}, \mathfrak{P}^1, \mathfrak{P}^2, \epsilon^1, \epsilon^2$. *A matrix* κ^{12} *is modular as to* $\epsilon^1\ \epsilon^2$ *if and only if* κ^{12} *is by rows of* $\overline{\mathfrak{M}}^2$ *and* $J^2\kappa^{12}\mu^2$ *is modular as to* ϵ^1 *for every* μ^2.

To prove the theorem, we make use of the fact that κ^{12} is modular as to $\epsilon^1\ \epsilon^2$ if and only if the following condition holds:

(M) κ^{12} *is by rows of* $\overline{\mathfrak{M}}^2$, *and* $J^2\kappa^{12}\kappa^{*21}$ *is modular as to* $\epsilon^1\ \epsilon^1$.

This is Theorem (46.9) in G.A., with the omission of the redundant condition that κ^{12} is by columns accordant as to ϵ^1. (For when κ^{12} satisfies the conditions (M), κ^{12} is by columns A^1. To prove this, we note that $J^2\kappa^{12}\kappa^{*21}$ is A^{11} by Theorem (46.65) in G.A. Consequently, when $S_\sigma^1\epsilon^1\alpha^1 = 0^1$, then $J^2(S_\sigma^1\bar{\alpha}^1\kappa^{12}, S_\sigma^1\kappa^{*21}\alpha^1) = S_\sigma^1 S_\sigma^1\bar{\alpha}^1 J^2\kappa^{12}\kappa^{*21}\alpha^1 = 0$, which implies that $S_\sigma^1\bar{\alpha}^1\kappa^{12} = 0^2$, since J^2 is proper. Thus κ^{12} is by columns A^1.) Consequently, it suffices to prove the following statement: *When* κ^{12} *is by rows of* $\overline{\mathfrak{M}}^2$, *the matrix* $J^2\kappa^{12}\kappa^{*21}$ *is modular as to* $\epsilon^1\ \epsilon^1$ *if and only if* $J^2\kappa^{21}\mu^2$ *is modular as to* ϵ^1 *for every* μ^2.

Using the notation introduced by E. H. Moore in his study of generalized Fourier theory, we denote the positive hermitian matrix $J^2\kappa^{21}\kappa^{*21}$ by $\epsilon_{\kappa*}^1$. It was shown by Moore that the space of vectors modular as to $\epsilon_{\kappa*}^1$ is equal[8] to the space of vectors $J^2\kappa^{12}\mu^2$ for all μ^2 in $\mathfrak{M}^2$. When κ^{12} is assumed to be by rows of $\overline{\mathfrak{M}}^2$, the assertion that $J^2\kappa^{12}\mu^2$ is modular as to ϵ^1 for every μ^2 is equivalent to the assertion that every vector modular as to $\epsilon_{\kappa*}^1$ is modular as to ϵ^1. By Theorem I, the latter assertion is valid if and only if $\epsilon_{\kappa*}^1$ is modular as to $\epsilon^1\ \epsilon^1$. This proves the theorem.

The basis stated in the preceding theorem is assumed for the following corollary:

COROLLARY. *Suppose that* κ^{12} *is by rows of* $\overline{\mathfrak{M}}^2$. *Then* κ^{12} *is modular as to* $\epsilon^1\ \epsilon^2$ *if and only if every vector modular as to* $J^2\kappa^{12}\kappa^{*21}$ *is modular as to* ϵ^1.

The transitivity for accordance and finiteness is stated in the following theorem:

[8] Moore, G. A., I, p. 22.

THEOREM IV. *Consider the basis* $\mathfrak{A}$, $\mathfrak{P}$, ϵ *and let* ϵ_0 *be a positive hermitian matrix. Then*

(a) *every vector accordant as to* ϵ_0 *is accordant as to* ϵ *if and only if* ϵ_0 *is accordant as to* ϵ ϵ;

(b) *every vector finite as to* ϵ_0 *is finite as to* ϵ *if and only if* ϵ_0 *is of type* $F\overline{F}$.

In part (a), if every vector accordant as to ϵ_0 is accordant as to ϵ, then ϵ_0, being of type $A_0\overline{A}_0$, is of type $A\overline{A}$. By Theorem (46.5) in G.A., ϵ_0 is accordant as to ϵ ϵ. Conversely every vector ξ accordant as to ϵ_0 satisfies the relation $\xi = J_0\epsilon_0\xi = L_\sigma J_0\epsilon_0\xi_\sigma$. Now $J_0\epsilon_0\xi_\sigma$, being a finite (right) linear combination of the columns of ϵ_0, is a vector accordant as to ϵ for every σ. By Theorem (48.2) in G.A., ξ is accordant as to ϵ. Part (b) is an immediate consequence of the definition of finiteness.

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