A NOTE ON HERMITIAN FORMS[1]

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In this note we effect a reduction of the theory of hermitian forms of two particular types (coefficients in a quadratic field or in a quaternion algebra with the usual anti-automorphism) to that of quadratic forms. The main theorem (§2) enables us to apply directly the known results on quadratic forms. This is illustrated in the discussion in §3 of a number of special cases.

Let Φ be an arbitrary quasi-field of characteristic different from 2 in which an involutorial anti-automorphism $\alpha\to\bar{\alpha}$ is defined. For the present we do not exclude the cases where Φ is commutative and $\bar{\alpha}\equiv\alpha$ or Φ is a quadratic field with $\alpha\to\bar{\alpha}$ as its automorphism. Suppose $\mathfrak{R}$ is an n-dimensional vector space over Φ. We define a bilinear form (x, y) as a function of pairs of vectors with values in Φ, such that

$$(1)\qquad \begin{aligned} (x_1 + x_2, y) &= (x_1, y) + (x_2, y), & (x, y_1 + y_2) &= (x, y_1) + (x, y_2),\\ (x, y\alpha) &= (x, y)\alpha, & (x\alpha, y) &= \bar{\alpha}(x, y), \end{aligned}$$

for all x, y in $\mathfrak{R}$ and α in Φ. If $x_1, x_2, \cdots, x_n$ is a basis for $\mathfrak{R}$ and $(x_i, x_j)=\alpha_{ij}$, the matrix $A=(\alpha_{ij})$ is called the matrix of (x, y) relative to this basis. By (1) it determines (x, y) as $\sum\bar{\xi}_i\alpha_{ij}\eta_j$, if $x=\sum x_i\xi_i$ and $y=\sum x_i\eta_i$. If $y_1, y_2, \cdots, y_n$ where $y_i=\sum x_j\rho_{ji}$ is a second basis for $\mathfrak{R}$ where $R=(\rho_{ij})$ is nonsingular, the matrix of (x, y) relative to this basis is $\bar{R}'AR$. We call A and $\bar{R}'AR$ cogredient. The form (x, y) is hermitian (skew-hermitian), if $(y, x)=\overline{(x, y)}$ $((y, x)=-\overline{(x, y)})$. This is equivalent to the condition $\bar{A}'=A$ $(\bar{A}'=-A)$.

It is readily seen that we may pass from the basis y_i to the x's by a sequence of substitutions of the following two types:

I. $y_i\to y_i$, $(i\neq r)$, $y_r\to y_r+y_s\theta$, $(s\neq r)$.

II. $y_i\to y_i$, $(i\neq r)$, $y_r\to y_r\theta$, $(\theta\neq 0)$.

It follows that we may pass from a matrix to any other matrix cogredient to it by a sequence of transformations of the corresponding types:

I. Addition of the sth column multiplied on the right by θ to the rth together with addition of the sth row multiplied on the left by $\bar{\theta}$ to the rth.

II. Multiplication of the rth column on the right by $\theta\neq 0$ together with multiplication of the rth row on the left by $\bar{\theta}$.

We showed in an earlier paper that any hermitian form or skew-

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hermitian form with $\bar{\alpha} \neq \alpha$ has a matrix in diagonal form; that is, there is a basis $u_1, u_2, \cdots, u_n$ for $\mathfrak{R}$ such that[2] $(u_i, u_j)=0$, if $i \neq j$. We call a basis of this type orthogonal and u, v orthogonal, if $(u, v)=0$. If $(u_i, u_i)=\beta_i \neq 0$ for $i \leqq r$ and $(u_i, u_i)=0$ for $i>r$, we obtain the diagonal matrix

$$(2) \qquad [\beta_1, \beta_2, \cdots, \beta_r, 0, \cdots, 0]$$

for our form.[3] The element β_1 may be taken to be any nonzero element represented by the form, that is, any element for which a u_1 exists such that $(u_1, u_1)=\beta_1$, β_2 is any element represented by (x, y) restricted to the space of vectors orthogonal to u_1, and so on. We note also that β_i may be replaced by $\bar{\gamma}_i\beta_i\gamma_i$, $(\gamma_i \neq 0)$.

The space $\mathfrak{R}_0$ generated by $u_{r+1}, u_{r+2}, \cdots, u_n$ may be characterized as the totality of vectors z, such that $(x, z)=0$ for all x. The space $\mathfrak{R}_1$ generated by $u_1, u_2, \cdots, u_r$ satisfies the condition $\mathfrak{R}=\mathfrak{R}_0+\mathfrak{R}_1$, $\mathfrak{R}_0 \cap \mathfrak{R}_1=0$. If $\mathfrak{R}_2$ is a second space of this sort, it has a basis of the form u_i+z_i, $(i=1, \cdots, r)$, and hence the matrices of (x, y) in $\mathfrak{R}_1$ and in $\mathfrak{R}_2$ are cogredient. We may therefore restrict our attention to nondegenerate forms ($\mathfrak{R}_0=0$) and shall do so in the remainder of this note.

Two nondegenerate forms $(x, y)_1$ and $(x, y)_2$ in $\mathfrak{R}$ and $\mathfrak{R}'$ respectively are *cogredient* if there is a (1-1) correspondence $x \rightarrow x'$ between $\mathfrak{R}$ and $\mathfrak{R}'$ such that[4] $(x, y)_1=(x', y')_2$. It follows that

$$(x', (y_1 + y_2)')_2 = (x', y_1' + y_2')_2$$

and hence that $(y_1+y_2)'=y_1'+y_2'$. Similarly $(y\alpha)'=y'\alpha$ and so $x \rightarrow x'$ is a linear transformation and $\mathfrak{R}$ and $\mathfrak{R}'$ have the same dimensionality. If $x_1, x_2, \cdots, x_n$ is a basis for $\mathfrak{R}$, then $x_1', x_2', \cdots, x_n'$ is one for $\mathfrak{R}'$. The matrix of $(x, y)_1$ relative to the first basis is the same as that of $(x', y')_2$ relative to the second. Hence the matrices of $(x, y)_1$ and $(x', y')_2$ relative to any bases are cogredient and conversely cogredience of the matrices implies that of the forms.

We shall suppose from now on that Φ is either a quadratic field $\Phi_0(i)$, $i^2=-\lambda$ and $\bar{\alpha}=\alpha_0-i\alpha_1$ for $\alpha=\alpha_0+i\alpha_1$ or that $\Phi=\Phi_0(i, j)$ is a quaternion algebra in which $i^2=-\lambda$, $j^2=-\mu$, $k=ij=-ji$ and $\bar{\alpha}=\alpha_0-i\alpha_1-j\alpha_2-k\alpha_3$ for $\alpha=\alpha_0+i\alpha_1+j\alpha_2+k\alpha_3$. We suppose also that (x, y) is hermitian. Then $(x, x) \in \Phi_0$ and any β in (2) may be replaced

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[2] *Simple Lie algebras over a field of characteristic zero*, Duke Mathematical Journal, vol. 4 (1938), p. 542.

[3] The above notation for diagonal matrices will be used throughout this note.

[4] $\mathfrak{R}$ and $\mathfrak{R}'$ have the same quasi-field and anti-automorphism.

by $\beta N(\gamma)$, $N(\gamma)=\bar{\gamma}\gamma$. Let Φ_0' be the multiplicative group of nonzero elements in Φ_0, Φ_0^* the subgroup of norms, and $\Gamma=\Phi_0'/\Phi_0^*$. A determinant for any hermitian matrix A has been defined by E. H. Moore.[5] We recall that, if a matrix B has the form (2) with $r=n$, then $\det B=\beta_1\beta_2\cdots\beta_n$ and, if $A=\bar{R}'BR$, $\det A=N(\rho)\det B$. Thus the coset of $\det A$ in Γ is an invariant of the class of matrices cogredient to A (or an invariant of the form). We shall call this coset the discriminant of A (or of the form).

$\mathfrak{R}$ may be regarded as a vector space of $2n$ or $4n$ dimensions over Φ_0 and

$$\{x, y\} = (1/2)[(x, y) + (y, x)] = (1/2)\operatorname{tr}(x, y) \tag{3}$$

is a symmetric form in $\mathfrak{R}$ over Φ_0. The symmetric form $\{x, y\}$ satisfies the special condition

$$\{x\alpha, y\alpha\} = \{x, y\}N(\alpha), \tag{4}$$

whence

$$\{x\bar{\alpha}, y\} = \{x\bar{\alpha}, y\bar{\alpha}^{-1}\bar{\alpha}\} = \{x, y\bar{\alpha}^{-1}\}N(\alpha) = \{x, y\alpha\}.$$

Hence, if $\bar{\alpha}=-\alpha$, $\{x, x\alpha\}=-\{x\alpha, x\}=0$. Conversely, if $\{x, y\}$ is any symmetric bilinear form in $\mathfrak{R}$ over Φ_0 such that (4) holds, (x, y) defined by

$$(x, y) = \begin{cases} \{x, y\} - (i/\lambda)\{x, yi\}, & \text{if } \Phi = \Phi_0(i), \\ \{x, y\} - (i/\lambda)\{x, yi\} - (j/\mu)\{x, yj\} & \\ \qquad - (k/\lambda\mu)\{x, yk\}, & \text{if } \Phi = \Phi_0(i, j), \end{cases} \tag{5}$$

is hermitian in $\mathfrak{R}$ over Φ. The relation between (x, y) and $\{x, y\}$ is a reciprocal one and $\{x, y\}$ is nondegenerate if (x, y) is.[6]

Evidently, if $(x, y)_1$ in $\mathfrak{R}$ over Φ and $(x', y')_2$ in $\mathfrak{R}'$ over Φ are cogredient, then $\{x, y\}_1$ and $\{x', y'\}_2$ are cogredient also. Suppose now that $\{x, y\}_1$ and $\{x', y'\}_2$ are cogredient. Then we have u_1 and u_1', such that[7] $(u_1, u_1)_1=\{u_1, u_1\}_1=\{u_1', u_1'\}_2=(u_1', u_1')_2=\beta_1\neq 0$. Let $\mathfrak{R}_1$ and $\mathfrak{R}_1'$ respectively denote the spaces of vectors orthogonal to u_1 and u_1' relative to $(x, y)_1$ and $(x', y')_2$. The space $\mathfrak{R}_1$ may also be characterized as the set of vectors orthogonal to u_1, u_1i if $\Phi=\Phi_0(i)$, or to u_1, u_1i, u_1j, u_1k, if $\Phi=\Phi_0(i, j)$, with respect to $\{x, y\}_1$. A similar

[5] *General Analysis*, I, American Philosophical Society Publication, Philadelphia, 1935.

[6] We make use of the relation $a=\alpha_0+i\alpha_1+j\alpha_2+k\alpha_3=(1/2)[\operatorname{tr} a-(i/\lambda)\operatorname{tr} ai-(j/\mu)\operatorname{tr} aj-(k/\lambda\mu)\operatorname{tr} ak]$.

[7] There exists a vector u_1 such that $(u_1, u_1)\neq 0$. Cf. Jacobson, loc. cit.

characterization holds for $\mathfrak{R}_1'$. The matrix of $\{x, y\}_1$ relative to u_1, $u_1 i$ or $u_1, u_1 i, u_1 j, u_1 k$ and of $\{x', y'\}_2$ relative to $u_1', u_1' i$ or $u_1', u_1' i, u_1' j, u_1' k$ is

$$[\beta_1, \lambda\beta_1] \quad \text{or} \quad [\beta_1, \lambda\beta_1, \mu\beta_1, \lambda\mu\beta_1]. \tag{6}$$

Hence it follows from a theorem of Witt[8] that $\{x, y\}_1$ and $\{x', y'\}_2$ are cogredient when restricted to $\mathfrak{R}_1$ and $\mathfrak{R}_1'$. By induction $(x, y)_1$ and $(x', y')_2$ are cogredient. Thus we have proved the following theorem:

THEOREM. *A necessary and sufficient condition that two hermitian forms $(x, y)_1$ and $(x, y)_2$ be cogredient is that the corresponding symmetric forms $\{x, y\}_1$ and $\{x, y\}_2$ be cogredient.*

If $u_1, u_2, \cdots, u_n$ is an orthogonal basis, $(u_i, u_i) = \beta_i$, then $u_1, u_1 i, u_2, u_2 i, \cdots, u_n, u_n i$ or $u_1, u_1 i, u_1 j, u_1 k, \cdots, u_n, u_n i, u_n j, u_n k$ is an orthogonal basis for $\mathfrak{R}$ over Φ relative to $\{x, y\}$ and the corresponding matrix, where B_i is as in (6), is

$$[B_1, B_2, \cdots, B_n]. \tag{7}$$

We consider now some special cases:

(1) $\Phi_0(i)$, *where Φ_0 is a field in which every nondegenerate symmetric form in* 5 *or more variables is a null-form.* Examples of such fields are (a) any p-adic field, (b) an algebraic function field of one variable over a finite constant field.[9] In these cases any nondegenerate symmetric form in 4 or more variables represents every $\alpha \neq 0$ in Φ_0. For, if $\{x, y\}$ represents 0, say $\{u, u\} = 0$, we choose v such that $\{u, v\} = \beta \neq 0$. Then $\{u\xi + v\eta, u\xi + v\eta\} = \eta(2\beta\xi + \gamma\eta), \gamma = \{v, v\}$ and the equation $\eta(2\beta\xi + \gamma\eta) = \alpha$ can be solved for ξ, η in Φ_0. If $\{x, y\}$ does not represent 0, we form the vector space of $(n+1)$ dimensions by adjoining z to $\mathfrak{R}$, and define $\{x\xi + z\eta, x\rho + z\sigma\} = \{x, x\}\xi\rho - \alpha\eta\sigma$. Since this form represents 0, we have $\{x, x\}\xi^2 - \alpha\eta^2 = 0$ for $\eta \neq 0$ since $\{x, x\} \neq 0$. Thus $\{x\xi\eta^{-1}, x\xi\eta^{-1}\} = \alpha$. It follows that any hermitian form in a space of 2 or more dimensions represents any α in Φ_0. Hence we may choose $\beta_1 = \beta_2 = \cdots = \beta_{n-1} = 1$ in (2). Thus two forms are cogredient, if, and only if, they have the same discriminant.

(2) $\Phi_0(i, j)$, *Φ_0 of the same type as in case* (1). Here we may take $\beta_1 = \cdots = \beta_n = 1$ and hence all nondegenerate forms are cogredient.

(3) $\Phi_0(i)$, *Φ_0 a real closed field.* Here we may suppose $\lambda = 1$ and we may suppose $\beta_1 = \cdots = \beta_p = 1, \beta_{p+1} = \cdots = \beta_n = -1$. For $\{x, y\}$ we

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[8] *Theorie der quadratischen Formen in beliebigen Körpern*, Journal für die reine und angewandte Mathematik, vol. 176 (1936–1937), p. 34.

[9] Witt, loc. cit., p. 40, and Albert, *Quadratic null forms over a function field*, Annals of Mathematics, (2), vol. 39 (1938), pp. 494–505.

obtain $2p$ values $+1$ and $(2n-2p)$ values -1 in the diagonal form. Since the signature is an invariant for bilinear forms it is invariant also for the hermitian form (x, y).

(4) $\Phi_0(i, j)$, *Φ_0 a real closed field.* The considerations are similar to case (3). We find that two nondegenerate hermitian forms are cogredient if and only if they have the same signatures.[10]

(5) $\Phi_0(i)$, *Φ_0 an algebraic number field.* As is well known, the symmetric forms $\{x, y\}_1$ and $\{x, y\}_2$ in $\mathfrak{R}$ over Φ_0 are cogredient, if, and only if, they are cogredient in every p-adic extension of Φ_0. Suppose first that p is a prime spot such that $(-\lambda/p)=1$, that is, $-\lambda$ is a square in the p-adic field[11] Φ_{0p}. Then the matrix B_i in (7) is cogredient in Φ_{0p} to $[\beta_i, -\beta_i]$ and hence also[12] to $[1, -1]$. Thus any two matrices of the form (7) are cogredient. If $(-\lambda/p)=-1$, $\Phi_{0p}(i)$ is a quadratic field over Φ_{0p}. Hence $\{x, y\}_1$ and $\{x, y\}_2$ are cogredient, if, and only if, $(x, y)_1$ and $(x, y)_2$ are cogredient in $\mathfrak{R}$ over $\Phi_{0p}(i)$. The condition for this is that the discriminants be the same when p is finite and the signatures be the same when p is infinite. Combining these results, we see that a necessary and sufficient condition that two nondegenerate hermitian forms in $\mathfrak{R}$ over Φ be cogredient is that they have the same discriminant and the same signature at the infinite prime spots for which λ is positive.[13]

(6) $\Phi_0(i, j)$, *Φ_0 an algebraic number field.* To obtain conditions for cogredience of $(x, y)_1$ and $(x, y)_2$ we again consider $\{x, y\}_1$ and $\{x, y\}_2$. Let p be a prime spot at which $((-\lambda, -\mu)/p)=1$, that is, $\Phi_{0p}(i, j)$ is a matrix algebra. Then either $-\lambda$ is a square in Φ_{0p} or $-\mu$ is a norm in $\Phi_{0p}(i)$. In the first case B_i is cogredient to $[\beta_i, -\beta_i, \mu\beta_i, -\mu\beta_i]$ and hence to $[1, -1, 1, -1]$. If $-\mu$ is a norm in $\Phi_{0p}(i)$, the bilinear form with matrix $[\beta_i, \lambda\beta_i, \mu\beta_i]$ represents 0 and hence is cogredient to $[1, -1, -\lambda\mu\beta_i]$, and again (6) is cogredient to (8). If p is a prime spot for which $\Phi_{0p}(i, j)$ is a division algebra, $(x, y)_1$ and $(x, y)_2$ are always cogredient, if p is finite, and these forms are cogredient for p infinite, if, and only if, they have the same signatures. Thus a necessary and sufficient condition that $(x, y)_1$ and $(x, y)_2$ in $\mathfrak{R}$ over Φ be cogredient is that these forms have the same signatures for all infinite prime spots for which $\Phi_{0p}(i, j)$ is a division algebra.

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[10] E. H. Moore, loc. cit., p. 193.

[11] See Witt and the references cited there to Hasse's papers.

[12] This is a consequence of Witt's theorem that any two symmetric forms in two variables which are nonsingular and represent 0 are cogredient (Witt, p. 34).

[13] Cf. Landherr, *Äquivalenz Hermitescher Formen über einem beliebigen algebraischen Zahlkörper*, Abhandlungen aus dem mathematischen Seminar der Hansischen Universität, vol. 11 (1936), pp. 245–248.