A METHOD FOR PROVING CERTAIN ABSTRACT GROUPS TO BE INFINITE[1]

H. S. M. COXETER

1. **Introduction.** I have stated elsewhere[2] that the group (3, 3, 4; 4), defined by

$$R^3 = S^3 = (RS)^4 = (R^{-1}S^{-1}RS)^4 = 1,$$

is infinite. This fact will now be proved by showing that there is a factor group of order $24n^4$ for every positive integer n.

We shall find a closely related group of order $48n^4$, satisfying the relations $S^3 = T^2 = (ST)^8 = (S^{-1}TST)^6 = 1$, which have been studied by Brahana;[3] but there is no overlapping, since his "subgroup H" is not invariant in our case, although there still is an abelian invariant subgroup of index 48. In fact, it was the search for such a subgroup that led to the simple treatment given here.

Section 7 is inserted for its intrinsic interest, and can be omitted without impairing the proof of the main result (§8).

2. **A group of order n^4.** Consider the direct product of two cyclic groups of order n. Since the defining relations $M_1^n = M_2^n = M_1^{-1}M_2^{-1}M_1M_2 = 1$ imply $(M_1M_2)^n = 1$, they may be put into the form[4]

$$M_1^n = M_2^n = M_3^n = M_1M_2M_3 = M_3M_2M_1 = 1. \tag{1}$$

Hence the direct product of four cyclic groups of order n is defined by

$$\begin{aligned} M_i^n = M_1M_2M_3 = M_3M_2M_1 = N_j^n = N_1N_2N_3 = N_3N_2N_1 = 1, \\ M_iN_j = N_jM_i, \qquad\qquad i, j = 1, 2, 3. \end{aligned} \tag{2}$$

3. **A group of order $4n^4$.** These relations continue to hold when M_i is replaced by N_i, and N_j by M_j^{-1}. We now enlarge the group of order n^4 by adjoining an operator A, of period four, which transforms it according to this automorphism. The extra relations that have to be added to (2) are

$$A^4 = 1, \qquad A^{-1}M_iA = N_i, \qquad A^{-1}N_jA = M_j^{-1}.$$

[1] Presented to the Society, September 6, 1938. The enumerative method described in the abstract (this Bulletin, 44-9-331) seems to be effective only in those cases where more orthodox methods are equally effective.

[2] Coxeter [**2**, p. 101, second footnote].

[3] Brahana [**1**].

[4] In the notation of Coxeter [**2**, p. 87], this is $(n, n, n; 1)$.

The enlarged group,[5] of order $4n^4$, may be put into a symmetrical form by defining $B=AM_3$, $C=N_2A$, and eliminating the M's and N's by means of the relations

$$B^{-1}C = M_3^{-1}M_2^{-1} = M_1, \quad C^{-1}A = A^{-1}N_2^{-1}A = M_2, \quad A^{-1}B = M_3,$$
$$BC^{-1} = N_3^{-1}N_2^{-1} = N_1, \quad CA^{-1} = N_2, \quad AB^{-1} = AM_3^{-1}A^{-1} = N_3.$$

The result is

$$(3) \quad \begin{aligned} A^4 = B^4 = C^4 = (BC)^2 = (CA)^2 = (AB)^2 = A^{-1}BC^{-1}AB^{-1}C \\ = (B^{-1}C)^n = (C^{-1}A)^n = (A^{-1}B)^n = 1. \end{aligned}$$

These relations imply

$$\begin{aligned} (A^2B^{-1}C)^2 &= (B^2C^{-1}A)^2 = (C^2A^{-1}B)^2 = (A^{-1}B^2C)^2 = (B^{-1}C^2A)^2 \\ &= (C^{-1}A^2B)^2 = (A^2B^2C^2)^2 = (ABC)^4 = 1. \end{aligned}$$

In detail,

$$\begin{aligned} (A^2B^{-1}C)^2 &= A^{-1}\cdot A^{-1}B^{-1}\cdot CA\cdot AB^{-1}C = A^{-1}\cdot BA\cdot A^{-1}C^{-1}\cdot AB^{-1}C \\ &= A^{-1}BC^{-1}AB^{-1}C = 1, \\ (A^{-1}B^2C)^2 &= A^{-1}B\cdot BC\cdot A^{-1}B^{-1}\cdot B^{-1}C = A^{-1}B\cdot C^{-1}B^{-1}\cdot BA\cdot B^{-1}C \\ &= A^{-1}BC^{-1}AB^{-1}C = 1, \\ (A^2B^2C^2)^2 &= A^2B\cdot BC\cdot CA^2B^2C^2 = A^2BC^{-1}\cdot B^{-1}CA^2\cdot B^2C^2 \\ &= CB^{-1}A^{-2}\cdot A^{-2}C^{-1}B\cdot B^2C^2 = C\cdot B^{-1}C^{-1}B^{-1}\cdot C^2 = C^4 = 1, \\ (ABC)^4 &= (AB\cdot CA\cdot BC)^2 = (B^{-1}A^{-1}\cdot A^{-1}C^{-1}\cdot C^{-1}B^{-1})^2 \\ &= (BC^2A^2B)^{-2} = 1. \end{aligned}$$

As one of the relations $M_i^n=1$ is superfluous in (1), so one of the consequent relations $(B^{-1}C)^n=(C^{-1}A)^n=(A^{-1}B)^n=1$ is superfluous in (3), say $(C^{-1}A)^n=1$. In terms of A, B, C and $(ABC)^{-1}$, (3) takes the form

$$(4) \quad \begin{aligned} A^4=B^4=C^4=D^4 = ABCD = (BC)^2 = (CA)^2 = (AB)^2 = (BD)^2 \\ = (A^{-1}B)^n = (B^{-1}C)^n = (C^{-1}D)^n = (D^{-1}A)^n = 1, \end{aligned}$$

implying

$$BC^{-1}AB^{-1}CA^{-1} = BC^{-1}\cdot AB\cdot B\cdot BC\cdot A^{-1} = (BC^{-1}B^{-1}A^{-1})^2 = (BD)^2 = 1.$$

Of course, the relations $(C^{-1}D)^n=(D^{-1}A)^n=1$ (inserted for the

[5] There is an intermediate group, of order $2n^4$, generated by A^2, AB, B^2, BC, C^2, CA, and defined by $T_1^2=T_2^2=T_3^2=T_4^2=T_5^2=T_6^2=T_1T_2T_3T_4T_5T_6=(T_1T_2)^n=(T_2T_3)^n=(T_3T_4)^n=(T_4T_5)^n=(T_5T_6)^n=(T_6T_1)^n=(T_iT_jT_k)^2=1$, $(i<j<k)$.

sake of symmetry) are superfluous; in fact, the rest of (4) implies $AB^{-1}\cdot A^{-1}C=A^{-1}C\cdot AB^{-1}$, whence $(AD^{-1})^n=(A^2BC)^n=(AB^{-1}\cdot A^{-1}C)^n =(AB^{-1})^n(A^{-1}C)^n=1$, and similarly $(D^{-1}C)^n=(ABC^2)^n=(AC^{-1}\cdot B^{-1}C)^n=(AC^{-1})^n(B^{-1}C)^n=1$.

4. **A group of order** $8n^4$. The relations (4) continue to hold when A, B, C and D are replaced by C^{-1}, D^{-1}, D^2A $(=DA^{-1}D^{-1})$ and BC^2 $(=C^{-1}B^{-1}C)$. For, BC, CA, AB, BD, $A^{-1}B$, $B^{-1}C$, $C^{-1}D$, $D^{-1}A$ are then replaced by DA, DB, DC, AC, CD^{-1}, $D^{-1}A$, BA^{-1}, $C^{-1}B$. Let $\mathfrak{P}$ denote this automorphism. The repeated automorphism $\mathfrak{P}^2$ replaces A, B, C, D by DAD^{-1}, $C^{-1}BC$, BCB^{-1}, $A^{-1}DA$, and so is equivalent to transformation by BC or DA. We now enlarge the group of order $4n^4$ by adjoining an operator P, whose square is BC, and which transforms the group according to the automorphism $\mathfrak{P}$. The extra relations that have to be added to (4) are

$$P^2 = BC,\ P^{-1}AP = C^{-1},\ P^{-1}BP = D^{-1},\ P^{-1}CP = D^2A,\ P^{-1}DP = BC^2.$$

Defining $Q=AP$, so that $QP^{-1}=A$, $PQ=B$, $Q^{-1}P=C$, $P^{-1}Q^{-1}=D$, we obtain the enlarged group, of order $4n^4$, in the form

$$\begin{aligned} P^4 = Q^4 = (PQ)^4 = (P^{-1}Q)^4 &= (P^{-1}Q^{-1}PQ)^2 \\ &= (PQ^{-1}PQ)^n = (P^{-1}QPQ)^n = 1. \end{aligned} \tag{5}$$

Since

$$\begin{aligned} (PQ^2)^2 &= PQ\cdot QP\cdot Q^2 = BD^{-1}AB = BAB\cdot CA\cdot B \\ &= A^{-1}\cdot A^{-1}C^{-1}\cdot B = (B^{-1}CA^2)^{-1}, \\ (P^2Q)^2 &= P^2\cdot QP\cdot PQ = BCD^{-1}B = B\cdot CA\cdot BCB \\ &= B\cdot A^{-1}C^{-1}\cdot C^{-1} = (C^2AB^{-1})^{-1}, \\ P^{-1}Q^2P^{-1} &= C^{-1}A, \end{aligned}$$

the relations (5) must imply $(PQ^2)^4=(P^2Q)^4=(P^2Q^2)^n=1$. In terms of P, Q and $(PQ)^{-1}$, we have, therefore,

$$\begin{aligned} O^4 = P^4 = Q^4 = OPQ = (QPO)^2 &= (P^{-1}Q)^4 = (Q^{-1}O)^4 \\ = (O^{-1}P)^4 = (P^2Q^2)^n &= (Q^2O^2)^n = (O^2P^2)^n = 1. \end{aligned} \tag{6}$$

5. **A group of order** $24n^4$. To this group of order $8n^4$ we adjoin an operator R, of period three, which transforms the three generators according to a cyclic permutation.[6] The substitution $O=RQR^{-1}$, $P=R^{-1}QR$ gives us the enlarged group, of order $24n^4$, in the form

[6] Compare Coxeter [2, p. 96].

(7) $Q^4 = R^3 = (QR)^3 = (Q^{-1}R)^6 = (Q^{-1}R^{-1}QR)^4 = (Q^2R^{-1}Q^2R)^n = 1.$

In terms of R and $(QR)^{-1}$, this becomes

(8) $R^3 = S^3 = (RS)^4 = (R^{-1}S)^6 = (R^{-1}S^{-1}RS)^4 = (R^{-1}SRS^{-1}RS)^n = 1.$

6. **A group of order** $48n^4$. Finally, to the group of order $24n^4$ we adjoin an involutory operator T which interchanges R and S, obtaining

$$\begin{aligned} S^3 = T^2 = (ST)^8 = (S^{-1}TST)^6 &= [(S^{-1}T)^2(ST)^2]^4 \\ &= [(ST)^4T]^{2n} = 1. \end{aligned} \tag{9}$$

For, if $T^2=1$ and $R=TST$, we have $R^{-1}S^{-1}RS=TS^{-1}TS^{-1}TSTS$ and $R^{-1}SRS^{-1}RS=(TS^{-1}TSTS)^2=(TST\cdot T\cdot STSTS)^2$.

In terms of ST and T, (9) takes the form

(10) $U^8 = T^2 = (UT)^3 = (U^{-1}TUT)^6 = (U^{-2}TU^2T)^4 = (U^4T)^{2n} = 1.$

7. **Other related groups.** Several further groups with simple defining relations can be derived from those obtained above. For instance, adjoining to (3) an operator V which cyclically permutes A, B, C, we obtain the group

(11) $V^3 = A^4 = (V^{-1}AVA)^2 = (VA^{-1}VA)^3 = (V^{-1}A^{-1}VA)^n = 1,$

of order $12n^4$, and we deduce that these relations imply $(VA)^{12} = (VA^2)^6 = 1$.

Again, adjoining to (4) an operator X which cyclically permutes A, B, C, D, we obtain

$$X^4 = A^4 = (XA)^4 = (X^2A)^4 = (X^{-1}AXA)^2 = (X^{-1}A^{-1}XA)^n = 1,$$

of order $16n^4$. In terms of X and XA, this becomes

(12) $X^4 = Y^4 = (XY)^4 = (X^{-1}Y)^4 = (X^2Y^2)^2 = (X^{-1}Y^{-1}XY)^n = 1.$

Concerning (5), it is natural to ask whether the periods of $PQ^{-1}PQ$ and $P^{-1}QPQ$ are inevitably equal. The rather surprising answer is, as we shall see, that by leaving one of them unrestricted we only double the order of the group. Since O, P, Q are interchangeable, this means that the group

(13) $P^4 = Q^4 = (PQ)^4 = (P^{-1}Q)^4 = (P^{-1}Q^{-1}PQ)^2 = (P^2Q^2)^n = 1$

is of order $16n^4$, like (12).

To build up such a group, we begin with the direct product of two cyclic groups of orders $2n$ and n (generated by M_2 and M_3), which can be written in a form resembling (1):

$$M_1^n = M_2^n, \qquad M_3^n = M_1M_2M_3 = M_3M_2M_1 = 1.$$

Instead of (2) we take the group

$$\begin{aligned} &M_1^n = M_2^n = N_1^n = N_2^n,\\ &M_3^n = M_1M_2M_3 = M_3M_2M_1 = N_3^n = N_1N_2N_3 = N_3N_2N_1 = 1,\\ &\qquad\qquad M_iN_j = N_jM_i, \qquad\qquad i, j = 1, 2, 3, \end{aligned}$$

whose order is $2n^4$ since its general operator can be expressed as

$$M_1^pM_2^qN_1^rN_2^sZ^t, \qquad 0 \leqq p, q, r, s < n;\ t = 0 \text{ or } 1,$$

where $Z = M_1^n$.

Instead of (3) and (4), we derive two equivalent definitions for a certain group of order $8n^4$: first[7]

$$\begin{aligned} A^4 = B^4 = C^4 = (BC)^2 = (CA)^2 = (AB)^2 = A^{-1}BC^{-1}AB^{-1}C\\ = (AC^{-1})^n(B^{-1}C)^n = (A^{-1}B)^n = 1, \end{aligned}$$

and second

$$\begin{aligned} A^4 = B^4 = C^4 = D^4 = ABCD = (BC)^2 = (CA)^2 = (AB)^2 = (BD)^2\\ = (A^{-1}B)^n = (C^{-1}D)^n = 1. \end{aligned}$$

Finally, instead of (5) we obtain the group, of order $16n^4$,

$$P^4 = Q^4 = (PQ)^4 = (P^{-1}Q)^4 = (P^{-1}Q^{-1}PQ)^2 = (PQ^{-1}PQ)^n = 1.$$

8. **Conclusions regarding infinite groups.** The consistency of (8) for all values of n shows that the group $(3, 3 | 4, 6; 4)$, defined by $R^3 = S^3 = (RS)^4 = (R^{-1}S)^6 = (R^{-1}S^{-1}RS)^4 = 1$, is infinite. The "larger" groups[8] $(3, 3 | 4, 6)$, $(3, 3, 4; 4)$ are infinite *a fortiori*. Similarly, (5) establishes infinite order for $(4, 4 | 4, 4; 2)$, and thence for[9] $(4, 4, 4; 2)$.

9. **Comparison with Brahana's groups.** The infinite group $(2, 3, 8; 6)$, of which (9) is a factor group, has been investigated by Brahana.[10] His operators T_1, T_2, T_3 are easily recognized in our factor group as BC, AB, D^2. Since $T_2T_3T_1 = CD \cdot D^2 \cdot DA = CA$, the subgroup

[7] These relations imply $(B^{-1}C)^n = (C^{-1}A)^n = (BC^{-1})^n = (CA^{-1})^n$. In detail, $(C^{-1}A)^{-n}(B^{-1}C)^n = (C^{-1}A)^{-n}(CA^{-1})^n = A^{-1}(AC^{-1})^{-n}(ACA^2)^nA = A^{-1}(B^{-1}C)^n(C^{-1}A)^nA = A^{-1}(B^{-1}A)^nA = 1$.

[8] Coxeter [**2**, pp. 86, 101].

[9] Coxeter [**2**, p. 97]. By the method of Coxeter [**2**, p. 90, §2.5], (12) establishes infinite order for $(4, 8 | 2, 4; 4)$. This raises an interesting question as to the finite or infinite order of $(4, 7 | 2, 4; 4)$.

[10] Brahana [**1**, p. 892].

$\{T_1, T_2, T_3\}$ is $\{BC, CA, AB\}$. This subgroup, being[11] $((n, n, n; 2))$, of order $2n^2$, is of index $24n^2$. It is not invariant,[12] since, if it were, its index would be just 24. Hence (9) is not one of the groups treated in Brahana's main theorems, but is a first step towards the "large undertaking" mentioned in his final paragraph.[13]

REFERENCES

1. H. R. Brahana, *On the groups generated by two operators of orders two and three whose product is of order eight*, American Journal of Mathematics, vol. 53 (1931), pp. 891–901.

2. H. S. M. Coxeter, *The abstract groups* $G^{m,n,p}$, Transactions of this Society, vol. 45 (1939), pp. 73–150.

UNIVERSITY OF TORONTO

[11] Coxeter [**2**, p. 143].
[12] Brahana [**1**, p. 892].
[13] Brahana [**1**, p. 901].