A THEOREM ON SURFACES[1]

ERNEST P. LANE

It is a fact which is familiar[2] to projective differential geometers that the non-rectilinear asymptotic curves on an analytic non-developable ruled surface, not a quadric, in ordinary three-dimensional space belong to linear complexes if, and only if, the ruled surface belongs to a linear congruence, that is, if, and only if, the ruled surface has two distinct or coincident rectilinear directrices. It is furthermore known[3] that in this case the asymptotic curves on the ruled surface are projectively equivalent. It has also been demonstrated[4] that if the asymptotic curves on an analytic non-ruled surface S in ordinary space belong to linear complexes, then the asymptotic ruled surfaces of S, that is, the ruled surfaces composed of the tangents of the asymptotic curves of either family, constructed at the points of a fixed asymptotic curve of the other family, on S have rectilinear directrices and therefore are such that their asymptotic curves belong to linear complexes. The converse of this theorem is also true, as will be shown below. Moreover, it will be proved below, by the aid of some formulas computed[5] by MacQueen and the author, that in this case the asymptotic curves of one family on the surface S are projectively equivalent, as are also the asymptotic curves of the other family on S.

In attempting to determine whether the asymptotic curves on the asymptotic ruled surfaces of a non-ruled surface S are twisted cubics if the asymptotic curves on the surface S are twisted cubics, the author answered this question in the affirmative and discovered the following general theorem, which seems to have escaped notice hitherto and which it is the purpose of this note to put on record and demonstrate:

If the asymptotic curves on an analytic non-ruled surface S in ordinary space belong to linear complexes, then the asymptotic curves of each family on S are projectively equivalent, not only to each other, but also

---

[1] Presented to the Society, December 2, 1939.

[2] A. Peter, *Die Flächen deren Haupttangentenkurven linearen Komplexen angehören*, Leipzig, dissertation, 1895.

[3] C. T. Sullivan, *Properties of surfaces whose asymptotic curves belong to linear complexes*, Transactions of this Society, vol. 15 (1914), pp. 167–196. See p. 171.

[4] Sullivan, loc. cit., p. 178; also Peter, loc. cit.

[5] E. P. Lane and M. L. MacQueen, *Surfaces whose asymptotic curves are twisted cubics*, American Journal of Mathematics, vol. 60 (1938), pp. 337–344. See p. 339.

to all the non-rectilinear asymptotic curves on all the asymptotic ruled surfaces of S which are composed of the tangents of the asymptotic curves of the other family on S, constructed at the points of the various asymptotic curves of the first family on S.

In order to prove this theorem it is convenient to collect here the notations and equations which form the analytic basis of a classical projective differential theory of non-ruled surfaces whose asymptotic curves belong to linear complexes. Let us consider in ordinary space an analytic non-ruled surface S whose parametric vector equation, referred to asymptotic parameters u, v, is

$$x = x(u, v). \tag{1}$$

The four coordinates x of a variable point x on the surface S satisfy two partial differential equations which can be reduced, by a suitably chosen transformation of proportionality factor, to Fubini's canonical form,

$$x_{uu} = px + \theta_u x_u + \beta x_v, \quad x_{vv} = qx + \gamma x_u + \theta_v x_v \quad (\theta = \log \beta\gamma), \tag{2}$$

subscripts indicating partial differentiation, and the coefficients being functions of u, v which satisfy three integrability conditions which need not be written here. It is known that if the parametric asymptotic curves on the surface S belong to linear complexes, the coefficients β, γ can, by a suitably chosen transformation of parameters, be specialized so that

$$\beta = \gamma = (U'V')^{1/2}/(U + V), \qquad U'V' \neq 0, \tag{3}$$

where U is an arbitrary function of u alone, and V of v alone, and the accent denotes differentiation with respect to the appropriate variables. Moreover, the coefficients p, q are in this case given by

$$\begin{aligned} p &= (1/2)(3l_{uu} - (3/2)l_u^2 - 3\beta l_v - U_1), \\ q &= (1/2)(3l_{vv} - (3/2)l_v^2 - 3\beta l_u - V_1), \end{aligned} \tag{4}$$

in which $l = \log \beta$ and U_1, V_1 are defined by

$$U_1 = (DU^2 + EU + F)/U', \quad V_1 = (DV^2 - EV + F)/V', \tag{5}$$

where D, E, F are arbitrary constants. The integrability conditions are identically satisfied by these expressions for the coefficients of equations (2).

It has been shown[6] that the fourth-order linear homogeneous dif-

[6] Lane and MacQueen, loc. cit., pp. 337–339.

ferential equation for the u-curve through a point x on the surface S has its seminvariants P_2, P_3, P_4 given by the formulas

$$(6)\qquad P_2 = (1/6)((1/2)S + U_1), \quad P_3 = (1/4)((1/2)S_1 + U_1'), \quad P_4 = (1/4)(U_1^2 + 3U_1''),$$

where S is the Schwarzian derivative of U, namely,

$$(7)\qquad S = (U''/U')' - (1/2)(U''/U')^2.$$

By means of the transformation of proportionality factor

$$(8)\qquad \bar{x} = \beta^{-3/2}x,$$

this differential equation for the u-curve through the point x can be written in the form

$$(9)\qquad \bar{x}_{uuuu} + 6P_2\bar{x}_{uu} + 4P_3\bar{x}_u + P_4\bar{x} = 0.$$

Since the coefficients of this equation are independent of v, it follows that the differential equation of all the u-curves on the surface S is this same equation and that these asymptotic u-curves are therefore projectively equivalent, and similarly for the v-curves on S, as previously stated.

It is now proposed to compute the differential equation of the same form for the non-rectilinear asymptotic curves on the asymptotic ruled surface R_v composed of v-tangents at the points of the u-curve through the point x on the surface S. Any point y on the v-tangent through the point x (except the point x itself) is given by the formula

$$(10)\qquad y = \beta^{-3/2}(hx + x_v),$$

where h is an arbitrary parameter. It is possible to determine h as a function of u, v so that the locus of the point y, when u varies and $v=$const., is an asymptotic curve on the ruled surface R_v. In fact, specializing a known[7] result, we find that the locus of the point y is an asymptotic curve on the surface R_v in case

$$(11)\qquad h_u = -(3/2)\beta^2.$$

But since $l_{uv}=\beta^2$, as can easily be verified, it follows that

$$(12)\qquad h = -(3/2)l_v + c + V_2,$$

where c is an arbitrary constant and V_2 is an arbitrary function of v

[7] Lane, *Projective Differential Geometry*, University of Chicago Press, 1932, p. 114, ex. 6.

alone. By the aid of this expression for h, and the transformation (8), we find

$$y = (c + V_2)\bar{x} + \bar{x}_v, \tag{13}$$

and besides differentiation of equation (9) with respect to v yields

$$(\bar{x}_v)_{uuuu} + 6P_2(\bar{x}_v)_{uu} + 4P_3(\bar{x}_v)_u + P_4(\bar{x}_v) = 0. \tag{14}$$

It is now evident that y also satisfies equation (9), identically in c and v. Therefore the differential equation for all the non-rectilinear asymptotic curves on all the asymptotic ruled surfaces R_v is the same as the differential equation (9) for the asymptotic u-curves on the surface S. It follows that all these curves are projectively equivalent. A similar argument can be made with u and v interchanged, and so the theorem is proved.

As a corollary, the theorem that the u-curves on the surface S belong to linear complexes if, and only if, the asymptotic curves on the surface R_v do so can be deduced at once, since these curves are now known to be projectively equivalent. In like manner, the theorem that the u-curves on the surface S are twisted cubics if, and only if, the asymptotic curves on the surface R_v are twisted cubics can be deduced.

UNIVERSITY OF CHICAGO