ON THE ABSOLUTE SUMMABILITY OF FOURIER SERIES. II[1]

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Bosanquet[2] has developed conditions for the absolute summability $C(\alpha)$ of a Fourier series. An immediate consequence of these conditions is that absolute summability is a local property for $\alpha > 1$. The purpose of this paper is to show by means of an example that absolute summability is not a local property for[3] $\alpha = 1$.

A Fourier series is absolutely summable $C(1)$ if $\sum_{m=1}^{\infty} |\sigma_m - \sigma_{m-1}| < \infty$. We have

$$\sigma_m - \sigma_{m-1} = \frac{1}{m+1}\sum_{n=0}^{m} S_n - \frac{1}{m}\sum_{n=0}^{m-1} S_n = \frac{\sigma_{m-1}}{m+1} - \frac{S_m}{m+1},$$

and, if $f(x)$ vanishes for $x \leqq x_0 > 0$, then at $x = 0$,

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{|\sigma_{m-1}|}{m+1} &\leqq \frac{1}{4\pi}\sum_{m=0}^{\infty} \frac{1}{m+1}\int_0^{\pi} |\phi(t)| \frac{\sin^2 (mt/2)}{m \sin^2 t/2} dt \\ &\leqq \frac{1}{4\pi}\sum_{m=0}^{\infty} \frac{1}{m^2 \sin^2 x_0/2}\int_0^{\pi} |\phi(t)| dt \\ &< \infty,\end{aligned}$$

so that it is only necessary to consider

$$\sum_{m=0}^{\infty} |A_m(f, x)| = \sum_{m=0}^{\infty} \frac{1}{2\pi(m+1)}\left|\int_0^{\pi} \phi(f, t) \frac{\sin (m + 1/2)t}{\sin t/2} dt\right|.$$

We define the functions

$$f_n(x) = \begin{cases}(n+1)\,|\sin x/2|, & \pi - \pi/3(n+1) \leqq |x| \leqq \pi, \\ 0, & \text{elsewhere}.\end{cases}$$

Then at $x = 0$, $\phi(f_n, t) = 2f_n(t)$ and, since

$$(-1)^m \sin (m + 1/2)t \geqq 1/2, \qquad \pi - \pi/3(m + 1/2) \leqq t \leqq \pi,$$

we have

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[1] Presented to the Society, April 8, 1939.

[2] L. S. Bosanquet, *The absolute summability of a Fourier series*, Proceedings of the London Mathematical Society, (2), vol. 41 (1936), pp. 517–528.

[3] This result has recently been proved by a different method by Bosanquet and Kestleman, *The absolute convergence of series and integrals*, ibid., vol. 45 (1939), pp. 88–97.

$$(1)\quad |A_m(f_n, 0)| > \frac{1}{4\pi(m+1)} \cdot \frac{(n+1)\pi}{3(n+1)} = \frac{1}{12(m+1)}, \qquad n > m,$$

and

$$A_m(f_n, 0) = \frac{(n+1)}{2\pi(m+1)(m+1/2)} \{\cos (m+1/2)t_0 - \cos (m+1/2)\pi\}, \quad t_0 = \pi - \pi/3(n+1),$$

so that

$$(2)\qquad |A_m(f_n, 0)| < (n+1)/\pi m^2.$$

By (1) it is possible to choose a sequence of integers $\{n_i\}$ in such a way that

$$(3)\qquad \sum_{m=0}^{n_i-1} |A_m(f_{n_i}, 0)| > \sum_{m=0}^{n_i-1} \frac{1}{24(m+1)} > 2^i.$$

The function $f(x)$ is then defined as $f(x) = \sum_{i=0}^{\infty} 2^{-i} f_{n_i}(x)$. For this function

$$\begin{aligned}\sum_{m=0}^{\infty} |A_m(f, 0)| &= \sum_{m=0}^{\infty} \left| \sum_{i=0}^{\infty} 2^{-i} A_m(f_{n_i}, 0) \right| \\ &\geqq \sum_{m=0}^{\infty} \sum_{n_i > m} 2^{-i} |A_m(f_{n_i}, 0)| \\ &\quad - \sum_{m=0}^{\infty} \sum_{n_i \leqq m} 2^{-i} |A_m(f_{n_i}, 0)|,\end{aligned}$$

since $|A_m(f_{n_i}, 0)| = (-1)^m A_m(f_{n_i}, 0)$ for $n_i > m$. By (2)

$$\begin{aligned}\sum_{m=0}^{\infty} \sum_{n_i \leqq m} 2^{-i} |A_m(f_{n_i}, 0)| &= \sum_{i=0}^{\infty} 2^{-i} \sum_{m \geqq n_i} |A_m(f_{n_i}, 0)| \\ &\leqq \sum_{i=0}^{\infty} 2^{-i} \sum_{m \geqq n_i} \frac{n_i}{m^2} < 2 \sum_{i=0}^{\infty} 2^{-i} = 4,\end{aligned}$$

and by (3)

$$\begin{aligned}\sum_{m=0}^{\infty} \sum_{n_i > m} 2^{-i} |A_m(f_{n_i}, 0)| &= \sum_{i=0}^{\infty} 2^{-i} \sum_{m < n_i} |A_m(f_{n_i}, 0)| \\ &> \sum_{i=0}^{\infty} 2^{-i} 2^i = \infty,\end{aligned}$$

so that $\sum_{m=}^{\infty} |A_m(f, 0)| = \infty$. It remains to show that $f(x) \subset L$ which is easily seen since

$$\int_{-\pi}^{\pi} | f(x) | \, dx = \sum_{i=0}^{\infty} 2^{-i} \int_{-\pi}^{\pi} | f_{n_i}(x) | \, dx$$

$$\leqq \sum_{i=0}^{\infty} 2^{-i} 2(n+1) \frac{\pi}{3(n+1)} < \infty .$$

We notice that, since this function vanishes in the neighborhood of the origin, it coincides with a function having an absolutely summable Fourier series in the neighborhood of the origin, and therefore absolute summability $C(1)$ is not a local property.

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COMPLETE REDUCIBILITY OF FORMS[1]

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1. **Introduction.** We shall say that F is a form in r *essential variables* with respect to a field K if F cannot be brought by means of a non-singular linear transformation in the field K to a form with less variables. Let F be a form of degree p written as $a_{ij\cdots k}x_i x_j \cdots x_k$, $(i, j, \cdots, k = 1, 2, \cdots, n)$. We arrange the coefficients of F in a matrix A whose n^{p-1} columns are of the form

$$\left\| \begin{matrix} a_{1j\cdots k} \\ a_{2j\cdots k} \\ \vdots \\ a_{nj\cdots k} \end{matrix} \right\| .$$

The index i is associated with the rows of A and the $p-1$ indices $j, \cdots, k$ are associated with the columns of A. We assume that the coefficients in F are so chosen that A is *symmetric* in the sense that the value of an element $a_{ij\cdots k}$ is unchanged under permutation of the subscripts. It can be shown[2] that F is *a form in r essential variables if and only if the rank of A is r.*

A form F is said to be *completely reducible* in a field K if F splits

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[1] Presented to the Society, April 7, 1939.

[2] Oldenburger, *Composition and rank of n-way matrices and multilinear forms,* Annals of Mathematics, (2), vol. 35 (1934), pp. 622–653.