

BOOK REVIEWS

General Analysis. Part 2. The Fundamental Notions of General Analysis. By Eliakim Hastings Moore. (Memoirs of the American Philosophical Society, vol. 1, part 2.) Philadelphia, American Philosophical Society, 1939. 6+255 pp.

The first part of this work, embodying the contributions of the late E. H. Moore on general analysis, was reviewed in this Bulletin, vol. 42 (1936), pp. 465–468, by C. C. MacDuffee.¹ The first part was devoted in the main to considerations of an algebraical character, no notion of continuity or limit being involved. While a general variable appears in the considerations, its generality is only incidental in that it is usually permitted to assume at most a finite number of distinct values. This second part begins a study from the point of view of analysis, in that limits notions play a substantial role. It is general analysis in the sense that a general unconditioned variable is involved.

The basis of this part includes a number system \mathfrak{A} of the type used at the end of the first part, with a continuity axiom added, namely, the existence of a greatest lower bound for nonvacuous sets of positive numbers. The resulting number system is then shown to be isomorphic to either the real number system, the complex number system, or the system of quaternions. There is assumed a general range \mathfrak{B} unconditioned. Vectors enter as functions on \mathfrak{B} to \mathfrak{A} , matrices as functions on $\mathfrak{B}^1\mathfrak{B}^2$ to \mathfrak{A} , and so on. While recent developments in linear functional analysis usually postulate a vector without considering it as a function on some range, the assumption made here, aside from its inherent advantages, has additional justification in that virtually all examples of the notion of vector have the character of a function.

The first chapter of this part (Chapter IV of the complete work) is devoted to an exposition of the notion of general limit. The general limit is a natural generalization of the notion of limit of a sequence, the positive integers being replaced by a general set \mathfrak{L} of elements l , on which there is given a relation R on pairs of elements which is transitive, that is, such that $l_1 R l_2$ and $l_2 R l_3$ imply $l_1 R l_3$, and compositive or semiordered, that is, l_1 and l_2 imply the existence of an l

¹ A review of this same part by the present reviewer appeared in *Zentralblatt für Mathematik*, vol. 13 (1936), pp. 116–117. We take this opportunity to correct a misprint appearing in MacDuffee's review, namely, the definition of general reciprocal λ of a matrix κ in the second last line of page 467 should read $SS\lambda\kappa = \kappa$ instead of $SS\lambda\kappa = \kappa$.

such that $l R l_1$ and $l R l_2$. Then the definition $\lim_l a_l = a$ follows the usual formula: for every positive e , there exists an l_e such that $l R l_e$ implies $|a_l - a| < e$. All the well known limits of analysis are special cases of this limit, provided in the continuous limit $\lim_{x \rightarrow x_0} f(x) = b$ it is assumed that $x \neq x_0$. In this particular work, the special case most frequently used is that in which the element l of \mathfrak{L} consists of a finite number of points σ of a set \mathfrak{B} , and the relation $\sigma_1 R \sigma_2$ means that σ_1 contains all of the points of σ_2 . In the theory of integration this limit has demonstrated its usefulness in its interpretation of l as a partition of the fundamental set into permissible subsets, the relation R being the notion "finer than." A detailed exposition of the principal properties of this limit as well as a theory of double and iterated limits is given. An interesting notion presented in this chapter is that of general sum: $\sum_l a_l$. Essentially \sum_l is $\lim_\sigma \sum_\sigma a_l$, where $\sigma = l_1, \dots, l_k$. As might be expected, the general sum exists only if at most a denumerable number of the elements a_l are different from zero, but the interesting thing is that the general sum can exist if and only if the sum of the absolute values exists, which applied to sequences says that an infinite series converges absolutely if and only if it converges in the sense that σ spreads.

The next chapter (Chapter V) gives an exposition of the basic feature of this contribution to general analysis, namely, the important theorem that any positive hermitian matrix (that is, a function ϵ on $\mathfrak{B}\mathfrak{B}$ to \mathfrak{A} for which $\epsilon(p_1 p_2) = \bar{\epsilon}(p_2 p_1)$ and such that if p_1, \dots, p_k are any elements of \mathfrak{B} , and a_1, \dots, a_k , any numbers of \mathfrak{A} , then $\sum_i \sum_j \bar{a}_i \epsilon(p_i p_j) a_j \equiv S_\sigma S_\sigma \bar{\alpha} \epsilon \alpha \geq 0$) defines a normed linear vector space, which has an inner product, or hermitian form, that is, is a general unitary space. The process of definition involves carrying over the notion of limited or modular matrix to limited or modular vector, and replacing the Kronecker δ by the positive hermitian matrix ϵ ; that is, we have that the vector ξ on \mathfrak{B} to \mathfrak{A} is modular relative to ϵ if and only if the $|S_\sigma \bar{\alpha} \xi|$ is bounded for all σ of \mathfrak{B} , and all α such that $S_\sigma S_\sigma \bar{\alpha} \epsilon \alpha \leq 1$. The least upper bound of these values when it exists is called the modulus of ξ , and the vectors for which this number is finite form the class \mathfrak{M} of modular vectors μ , which is a linear normed vector space. It is possible to obtain other forms of this notion of modularity. A vector ξ is modular if there exist scalars s , such that the matrix $\epsilon s^2 - \bar{\xi} \xi \equiv \epsilon(p_1 p_2) s^2 - \bar{\xi}(p_1) \xi(p_2)$ is positive, the smallest s for which this is so being the modulus of ξ ; or if γ_σ is the general reciprocal of $\epsilon_\sigma = \epsilon(p_i p_j)$, ($i, j = 1, \dots, n$), then ξ is modular if the expression $S_\sigma S_\sigma \bar{\xi} \gamma_\sigma \xi$ is bounded as a function of σ , and the least upper bound or limit in the σ sense of these scalars gives the square of the modulus.

If the matrix ϵ is positive definite, then this latter expression is well known as the quotient of two determinants, the denominator being that of $\epsilon(p_i p_j)$, ($i, j=1, \dots, n$), and the numerator being obtained by bordering this matrix with 0, $\xi(p_i)$ and $\bar{\xi}(p_j)$. It should be mentioned that the general reciprocal defined in Part I shows its usefulness here, in that it need not be assumed that the columns or rows of ϵ are linearly independent vectors. Moreover in addition to the boundedness property, the definition requires of any vector which is modular an accordant property; that is, if any set of rows (or columns) is linearly dependent, the corresponding places in the accordant vector satisfy the same dependence relation, that is, the rank of any submatrix of ϵ finite by rows is not changed by adding the accordant vector ξ to the columns.

The matrix ϵ gives rise to a bilinear integration process, namely, $\lim_{\sigma} S_{\sigma} S_{\sigma} \bar{\xi} \gamma_{\sigma} \eta = J \xi \eta$ (γ_{σ} the general reciprocal of ϵ_{σ}), effective between any two vectors which are modular, and resulting in an hermitian form, which gives us then a general not necessarily separable Hilbert space.

Special cases of the notion of modularity are interesting. It is obvious that when \mathfrak{P} is the set of positive integers, and ϵ is the Kronecker δ , then we obtain ordinary Hilbert space as the space of modular vectors. There are obvious generalizations of this to the case when \mathfrak{P} is any. If \mathfrak{P} is the continuous range $0 \leq p \leq 1$, and $\epsilon(p_1 p_2)$ is the greater of p_1 and p_2 , then modular vectors are absolutely continuous functions ξ such that $\xi(p) = \int_0^p \phi(q) dq$ where ϕ is of L^2 . If $\beta(E)$ is a positive additive set function, then the positive hermitian matrix $\beta(E_1 E_2)$ gives rise to Hellinger Radon integrals, for $p=2$. If a_{pn} are complex-valued vectors in ordinary Hilbert space for each p , then the positive hermitian matrix $\alpha_{pq} = \sum_n \bar{a}_{pn} a_{qn}$ gives rise to a modular space which plays an important role in the Schmidt theory of solutions of a system of infinitely many equations in Hilbert space.² If we think of \mathfrak{P} as consisting of the elements f, g, \dots of a Hilbert space, with hermitian form (f, g) , then the positive hermitian matrix (f, g) on $\mathfrak{P}\mathfrak{P}$ to \mathfrak{A} plays an important role in extending the results valid in a separable Hilbert space to the non-separable case.³ The process set up by Moore reveals itself in this setting as a natural generalization of the reduction of a denumerable system of vectors to an equivalent orthonormal system.

² See Riesz, *Les Systèmes d'Équations Linéaires à une Infinité d'Inconnus*, p. 66.

³ See Y. Y. Tseng, *The Characteristic Value Problem of Hermitian Functional Operators in a Non-Hilbertian Space*, University of Chicago thesis, 1933. Compare with F. Riesz, *Szeged Acta*, vol. 7 (1934-1935), pp. 34-38.

The balance of the chapter is taken up with convergence of modular vectors. We have a weak convergence (called mode 1) which involves convergence of μ_l for each p of \mathfrak{P} , and the ultimate boundedness of $M\mu_l$, and the usual strong convergence (called mode 2) equivalent to the convergence to zero of $M(\mu_l - \mu)$. In a later chapter it is proved that the former convergence is equivalent to the convergence of $L(\mu_l)$ to $L(\mu)$ for every linear continuous functional operation L on \mathfrak{M} to \mathfrak{A} , that is, the current definition of weak convergence. There is also introduced the notion of a sum of a set of vectors, involving a very strong type of convergence: $\sum_l M_l$.

Chapter VI is devoted to modular matrices. The definition of modular or limited matrix is, as might very well be expected from an analogy with vectors, as follows: $\kappa^{12} = \kappa(p^1 p^2)$ on $\mathfrak{P}^1 \mathfrak{P}^2$ to \mathfrak{A} is modular relative to the positive hermitian matrices ϵ^1 on $\mathfrak{P}^1 \mathfrak{P}^1$ to \mathfrak{A} and ϵ^2 on $\mathfrak{P}^2 \mathfrak{P}^2$ to \mathfrak{A} if $S_{\sigma^1} S_{\sigma^2} \bar{\alpha}^1 \kappa^{12} \alpha^2$ is bounded for all σ^1 and σ^2 and all α^1 and α^2 such that $S_{\sigma^1} S_{\sigma^1} \bar{\alpha}^1 \epsilon^1 \alpha^1 \leq 1$ and $S_{\sigma^2} S_{\sigma^2} \bar{\alpha}^2 \epsilon^2 \alpha^2 \leq 1$. There follows a thoroughgoing study of conditions of modularity, and questions of various kinds of convergence of modular matrices. Relative uniform convergence which plays such a prominent role in Moore's first general analysis enters naturally in the equivalence of strong convergence, that is, $\lim_l M(\kappa_l^{12} - \kappa^{12}) = 0$ with that of $J^1 J^2 \bar{\mu}^1 \kappa_l^{12} \mu^2$ to $J^1 J^2 \bar{\mu}^1 \kappa^{12} \mu^2$ uniform relative to $M\mu^1 M\mu^2$. For hermitian matrices it develops that a weak modularity, namely, the boundedness of $S_{\sigma} S_{\sigma} \bar{\alpha} \kappa \alpha$ for all σ and α such that $S_{\sigma} S_{\sigma} \bar{\alpha} \epsilon \alpha \leq 1$ is necessary and sufficient for modularity; further, if the number system \mathfrak{A} is not the real number system, this is also true for any matrix κ on $\mathfrak{P} \mathfrak{P}$ to \mathfrak{A} modular relative to ϵ, ϵ .

Chapter VII treats of linear functional operations, bilinear functional operations and linear functional transformations, especially those which are continuous or limited. In this setting every continuous linear functional operation is of the form $J\bar{\xi}\mu$, ξ modular, every limited or continuous bilinear functional operation determines and is determined by a modular matrix κ^{12} , namely, $\kappa^{12} = B(\epsilon^1, \epsilon^2)$ and $B(\bar{\mu}^1, \mu^2) = J^1 J^2 \bar{\mu}^1 \kappa^{12} \mu^2$, and every continuous linear transformation T on the modular class \mathfrak{M}^2 to \mathfrak{M}^1 is also determined by a limited matrix, namely, $\kappa^{12} = T\epsilon^2$ with $T\mu^2 = J^2 \kappa^{12} \mu^2$. Every completely continuous transformation, being defined as one in which weak convergence in \mathfrak{M}^2 implies strong convergence in \mathfrak{M}^1 , is the strong limit, that is, in the sense of the modulus of the transformations, of a transformation of a finite number of dimensions each of which is completely continuous, and the form of this approximation is the simple expression $J_{\sigma^2} \kappa^{12} \mu^2 = S_{\sigma^2} \kappa^{12} \gamma_{\sigma^2} \mu^2$, where γ_{σ^2} is the general reciprocal of ϵ_{σ^2} . Vari-

ous kinds of convergence of sets of linear, bilinear operations, and transformations are discussed. In many respects the last two chapters might be thought of as carrying over into the general setting the results of Hilbert and his followers on limited matrices and functional operations on ordinary Hilbert space.

The method of exposition in this second part follows closely the lines of the first part. Each chapter has an introduction giving an excellent survey of the material to be covered in the chapter, most theorems are stated not only in words, but also in the adaptation of the Peano symbolism introduced by Moore. To any one reading any considerable portion of this work, and consequently acquiring easily a familiarity with the symbolism used, this constant presentation of the same ideas in two forms unfortunately gets to be a little bit tiresome. The exposition is throughout very clear, very easily followed, and might even in some instances have assumed greater intelligence on the part of the reader. The reviewer was conscious of the paucity of references to the supporting literature, especially that current at the time when these developments of Moore were under way. While an isolationist point of view may have been justified at the time of development, the work would be enhanced historically and in comprehensibility if more frequent contacts with the literature were made available, and this would be in line with the spirit of E. H. Moore as the reviewer knew him thirty years ago.

To make an estimate of the value of this publication at this time is a little difficult. Much of it seems only historically worth while in the light of more recent developments in linear functional theory. The general limit has already shown its value in recent work. In the same way, the reviewer feels strongly that the notion of modularity is important, as well as the constructive procedure for hermitian operations on which Hilbert spaces are based. These two notions alone make this part worth while. Many of the results presented are basic to the parts of this publication to appear later, and so complete judgment must be deferred until these further developments are presented.

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It is to be understood that this is the first of two volumes on analysis and hence the author's aim is only to cover some of the traditional fundamentals of algebra and calculus.