ON FINITE ABELIAN p-GROUPS*

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Let p denote a prime number which is fixed throughout the course of the paper. As we shall deal only with abelian groups whose order is some power of p, the term group will be used exclusively to designate a group of this nature.

DEFINITION. *A subgroup $\mathfrak{H}$ of a group $\mathfrak{G}$ is said to be reciprocally cyclic if the factor group $\mathfrak{G}/\mathfrak{H}$ is cyclic.*

The letters k, m, n, r, x will be used to denote nonnegative integers and the symbol $(S_1, \cdots, S_n)$ to denote the group generated by the elements $S_1, \cdots, S_n$.

DEFINITION. *A set of elements $S_1, \cdots, S_n$ of the group $\mathfrak{G}$ is said to define a composition series for $\mathfrak{G}$ if the group series*

$$(S_1, \cdots, S_n), (S_1, \cdots, S_{n-1}), \cdots, (S_1, S_2), (S_1), (E)$$

is a composition series for the group $\mathfrak{G}$.

DEFINITION. *A set of elements $S_1, S_2, \cdots, S_n$ of the group $\mathfrak{G}$ is said to form a coverage system of the group $\mathfrak{G}$ if for any reciprocally cyclic subgroup $\mathfrak{H}$ of $\mathfrak{G}$ we may select elements $S_{m_1}, \cdots, S_{m_k}$ so that the group series*

$$(S_{m_1}, \cdots, S_{m_k}, \mathfrak{H}), (S_{m_1}, \cdots, S_{m_{k-1}}, \mathfrak{H}), \cdots, (S_{m_1}, \mathfrak{H}), \mathfrak{H}$$

is a composition series from $\mathfrak{G}$ to $\mathfrak{H}$.

A set of elements $S_1, \cdots, S_n$ which defines a composition series for a group $\mathfrak{G}$ also forms a coverage system for $\mathfrak{G}$. For let

$$(S_1, \cdots, S_n), (S_1, \cdots, S_{n-1}), \cdots, (S_1, S_2), (S_1), (E)$$

be the composition series of $\mathfrak{G}$ resulting from the element system $S_1, \cdots, S_n$ and $\mathfrak{H}$ any reciprocally cyclic subgroup of $\mathfrak{G}$; then the series

$$(S_1, \cdots, S_n, \mathfrak{H}), (S_1, \cdots, S_{n-1}, \mathfrak{H}), \cdots, (S_1, \mathfrak{H}), \mathfrak{H}$$

is a composition series from $\mathfrak{G}$ to $\mathfrak{H}$ with possible repetitions. These may be removed by starting with S_1 and striking out successively from left to right each member of the set $S_1, \cdots, S_n$ which is unnecessary for the generation of the groups of the above series. The

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elements remaining define a composition series from $\mathfrak{G}$ to $\mathfrak{H}$ which shows the set to be a coverage system for the group $\mathfrak{G}$.

This investigation deals with the converse of the above problem which may be stated as follows: $\mathfrak{G}$ is a group with composition series of length $n+1$ and with coverage system $S_1, \cdots, S_n$. Does it follow that this coverage system (after a possible rearrangement of order of the terms) defines a composition series for $\mathfrak{G}$? For groups of rank 1, that is, for cyclic groups the answer is clearly in the affirmative. The present paper shows the result to be true for groups of rank 2. The problem arises in the theory of numbers but is closely connected with ideas involved in the Jordan-Hölder theorem.

THEOREM A. *$\mathfrak{H}$ is any subgroup of the group $\mathfrak{G}$. The elements $S_1, \cdots, S_n$ form a coverage system of the group $\mathfrak{G}$. Then the restclasses defined by these elements in the factor group $\mathfrak{G}/\mathfrak{H}$ define a coverage system for this group.*

PROOF. Let $\overline{\mathfrak{F}}$ be a reciprocally cyclic subgroup of the factor group $\mathfrak{G}/\mathfrak{H}$. If $\mathfrak{F}$ be the maximum subgroup of the group $\mathfrak{G}$ which is built into $\overline{\mathfrak{F}}$ in the homomorphism $\mathfrak{G}\sim\mathfrak{G}/\mathfrak{H}$, it follows from the second isomorphism theorem that $\mathfrak{G}/\mathfrak{F}\cong\mathfrak{G}/\mathfrak{H}/\overline{\mathfrak{F}}$. This latter group is by hypothesis cyclic. Consequently $\mathfrak{F}$ is a reciprocally cyclic subgroup of $\mathfrak{G}$. Then by the definition of the coverage system, elements $S_{m_1}, S_{m_2}, \cdots, S_{m_k}$ exist so that the series

$$(S_{m_1}, S_{m_2}, \cdots, S_{m_k}, \mathfrak{F}), \cdots, (S_{m_1}, \mathfrak{F}), \mathfrak{F}$$

is a composition series from $\mathfrak{G}$ to $\mathfrak{F}$.

Let $\overline{S}_{m_1}, \overline{S}_{m_2}, \cdots, \overline{S}_{m_k}$ denote the restclasses of $\mathfrak{G}/\mathfrak{H}$ defined by the elements $S_{m_1}, S_{m_2}, \cdots, S_{m_k}$, respectively. Now in a homomorphism a composition series is built into a composition series with possible repetitions. Accordingly, as in the homorphism $\mathfrak{G}\sim\mathfrak{G}/\mathfrak{H}$, the series

$$(S_{m_1}, S_{m_2}, \cdots, S_{m_k}, \mathfrak{F}), \cdots, (S_{m_1}, \mathfrak{F}), \mathfrak{F}$$

is built into the series

$$(\overline{S}_{m_1}, \overline{S}_{m_2}, \cdots, \overline{S}_{m_k}, \overline{\mathfrak{F}}), \cdots, (\overline{S}_{m_1}, \overline{\mathfrak{F}}), \overline{\mathfrak{F}};$$

this second series is a composition series from $\mathfrak{G}/\mathfrak{H}$ to $\overline{\mathfrak{F}}$ with possible repetitions. Starting with the term $\overline{S}_{m_1}$ and proceeding from left to right, we successively remove every term of the set $\overline{S}_{m_1}, \overline{S}_{m_2}, \cdots, \overline{S}_{m_k}$ which is unnecessary for the generation of the groups of this series. After the repetitions have been thus removed, the series is a composition series from $\mathfrak{G}/\mathfrak{H}$ to $\overline{\mathfrak{F}}$ generated by the terms remaining of the set $\overline{S}_{m_1}, \overline{S}_{m_2}, \cdots, \overline{S}_{m_k}$. Thus the theorem is proved.

THEOREM B. *$\mathfrak{G}$ is a group for which the set $S_1, \cdots, S_n$ defines* (1) *a coverage system for $\mathfrak{G}$,* (2) *a composition series for $\mathfrak{G}$. S is an element of order p for which either $S=S_k$, or $S_k=SS_m^x$, $(p\nmid x, k>m)$. Then the restclasses defined by $S_1, \cdots, S_{k-1}, S_{k+1}, \cdots, S_n$ define* (1) *a coverage system for $\mathfrak{G}/(S)$,* (2) *a composition series for $\mathfrak{G}/(S)$.*

PROOF. Let us consider (1) first. By Theorem A the restclasses defined by $S_1, \cdots, S_n$ in the group $\mathfrak{G}/(S)$ form a coverage system. If $S=S_k$, the element S_k represents the unit restclass and so may be omitted from the coverage system. As for the second possibility, $S_k=SS_m^x$, we note that, as $p\nmid x$, S_m and S_m^x are equivalent as group generators, that is, we may always use S_m as a generator in place of S_m^x. In the factor group $\mathfrak{G}/(S)$, S_k and S_m^x represent the same restclass and consequently S_k may be omitted from the coverage system. Thus (1) is true. To prove (2) we use the above relations to show that the restclasses defined by $S_1, \cdots, S_{k-1}, S_{k+1}, \cdots, S_n$ define a composition series of $\mathfrak{G}/(S)$ with possible repetitions. As the length of the composition series of $\mathfrak{G}/(S)$ is n there can be no repetitions and so the result of the theorem is established.

The hypothesis of Theorems C and D is as follows. $S_1, \cdots, S_n$ is a system of elements, with S_1 the only element of order p, which defines a composition series for the noncyclic group $\mathfrak{G}$. The number a is the least subscript so that the group generated by the elements $S_1, \cdots, S_a$ is not cyclic; X is any element for which $X^p=E$ and $(S_{a-1}, X) = (S_{a-1}, S_a)$. Immediately from these relations follows

$$S_a = S_r^{x_1}X^{x_2}, \qquad p\nmid x_1,\ p\nmid x_2,\ 1 < r < a. \tag{1}$$

Let $T_1=S_{m_1}$, $T_2=S_{m_2}, \cdots, T_k=S_{m_k}$ be a greatest possible subset chosen from $S_2, \cdots, S_n$ so that $X, T_1, \cdots, T_k$ defines a composition series for the subgroup $\mathfrak{G}_X=(X, T_1, \cdots, T_k)$. If the subset is void, $\mathfrak{G}_X=(X)$.

THEOREM C. *If Y satisfy the same conditions as X with $(X)\neq(Y)$ and $\mathfrak{G}_Y$ be defined analogously to $\mathfrak{G}_X$, then $\mathfrak{G}_X \cap \mathfrak{G}_Y=(E)$.*

PROOF. We first show

$$S_1 \notin \mathfrak{G}_X. \tag{2}$$

If this were false and $S_1 \in \mathfrak{G}_X$, then $S_2 \in \mathfrak{G}_X$, for otherwise from $(S_2, S_1):(S_1)=p$ would follow $(\mathfrak{G}_X, S_2):\mathfrak{G}_X=p$ and $X, T_1, T_2, \cdots, T_k$ would not be a greatest possible subset. We could in turn argue similarly $S_3 \in \mathfrak{G}_X$ and so for all elements $S_2, \cdots, S_n$. Hence $\mathfrak{G}_X=\mathfrak{G}$ and thus $X, T_1, \cdots, T_k$ would define a composition series for $\mathfrak{G}$. Then, from the Jordan-Hölder theorem, $k=n-1$ and consequently $T_1, \cdots, T_k$ would be the complete set $S_2, \cdots, S_n$. Therefore, for S_r defined by (1), elements T_{m_1}, T_{m_2} would exist so that $T_{m_1}=S_r$, $T_{m_2}=S_a$. Now one of T_{m_1}, T_{m_2}, say T_{m_1}, must precede the other. Using (1) from Theorem B, it would then follow that $X, T_1, \cdots, T_{m_2-1}, T_{m_2+1}, \cdots, T_k$ would define a composition series for $\mathfrak{G}/(X)$ which is impossible as X is within the unit restclass. Therefore (2) must hold.

From the definition of a and the fact that $S_a^p \neq E$ follows $S_1 \varepsilon (S_x)$, $(1 \leqq x \leqq a)$. Therefore $S_x \varepsilon \mathfrak{G}_X$, $(1 \leqq x \leqq a)$, has as a consequence $S_1 \varepsilon \mathfrak{G}_X$. This being false, we have

$$S_x \notin \mathfrak{G}_X, \qquad 1 \leqq x \leqq a. \tag{3}$$

LEMMA. *For $S \varepsilon (X, T_1, \cdots, T_{k_1})$, $(k_1 \leqq k)$, a representation*

$$S = X^{r_0} S_{n_1}^{r_1} \cdots S_{n_{k_2}}^{r_{k_2}}, \qquad p \nmid r_1, p \nmid r_2, \cdots, p \nmid r_{k_2};\ n_1 < n_2 < \cdots < n_{k_2},$$

exists where $S_{n_1}, \cdots, S_{n_{k_2}}$ are members of the set $T_1, \cdots, T_{k_1}$.

PROOF OF LEMMA. If $S \varepsilon (X, T_1, \cdots, T_{k_1})$, integers $x_0, \cdots, x_{k_1}$, exist for which $S = X^{x_0} T_1^{x_1} \cdots T_{k_1}^{x_{k_1}}$. Let x_{k_3} be the first of these integers from the right for which $p \mid x_{k_3}$, $x_{k_3} \neq 0$. Now $k_3 > 0$, otherwise there is nothing to prove. As $(X, T_1, \cdots, T_{k_3}):(X, T_1, \cdots, T_{k_3-1}) = p$, $T_{k_3}^p \varepsilon (X, T_1, \cdots, T_{k_3-1})$; X may be regarded as T_0. Therefore an expression for $T_{k_3}^{x_{k_3}}$ exists in terms of $X, T_1, \cdots, T_{k_3-1}$. Substituting this expression for $T_{k_3}^{x_{k_3}}$ into the above representation for S, we obtain a new representation for S in which the exponent of T_{k_3} is zero and the exponents of $T_{k_3+1}, \cdots, T_{k_1}$ are the same as before. By successively repeating this process we ultimately arrive at an expression for S in which the exponents of $T_1, \cdots, T_{k_1}$ are either zero or prime to p. Omitting the factors with zero exponents, we now substitute for those remaining their values in the set $S_1, \cdots, S_n$, thus obtaining after a possible rearrangement of order the representation of S in the lemma.

Now let us assume the theorem false, that is, the meet $\mathfrak{G}_X \cap \mathfrak{G}_Y$ contains a non-unit element. From (1) and the analogous expression in Y we obtain $Y = S_1^{x_2} X^{x_3}$, $(p \nmid x_3)$. Then as $(X) \neq (Y)$, $p \nmid x_2$. From this, by applying (2) to $\mathfrak{G}_Y$, follows $X \notin \mathfrak{G}_Y$. Hence a least subscript k_4, $(1 \leqq k_4 \leqq k)$, exists so that $(X, T_1, \cdots, T_{k_4})$ and $\mathfrak{G}_Y$ have a common non-unit element. Let Z be any such element. As $Z^p \varepsilon (X, T_1, \cdots, T_{k_4-1})$, X being regarded as T_0, and $Z^p \varepsilon \mathfrak{G}_Y$, follows $Z^p = E$, because, by the minimum assumption, $(X, T_1, \cdots, T_{k_4-1}) \cap \mathfrak{G}_Y = (E)$. Therefore, each of $T_1, \cdots, T_{k_4}$ being by hypothesis of order greater than p,

$$T_m \notin \mathfrak{G}_Y, \qquad 1 \leqq m \leqq k_4. \tag{4}$$

Applying the lemma first in $\mathfrak{G}_X$ and then in $\mathfrak{G}_Y$

$$Z = X^{r_0} S_{n_1}^{r_1} \cdots S_{n_{k_5}}^{r_{k_5}}, \qquad n_1 < n_2 < \cdots < n_{k_5};\ p \nmid r_{k_5},$$

$$Z = Y^{s_0} S_{m_1}^{s_1} \cdots S_{m_{k_6}}^{s_{k_6}}, \qquad m_1 < m_2 < \cdots < m_{k_6};\ p \nmid s_{k_6},$$

where $S_{n_1}, \cdots, S_{n_{k_5}}$ are members of the set $T_1, \cdots, T_{k_4}$ and $S_{m_1}, \cdots, S_{m_{k_6}} \varepsilon\, \mathfrak{G}_Y$. Now n_{k_5} cannot reduce to zero for then $X \varepsilon (Z) \leqq \mathfrak{G}_Y$, contrary to what we have proved; moreover by (3), $n_{k_5} > a$; from (4), one of n_{k_5}, m_{k_6}, say n_{k_5}, is greater than the other and so $n_{k_5} - 1 \geqq m_{k_6}$. Then from the two expressions for Z and the fact that X, $Y \varepsilon (S_1, \cdots, S_a)$ follows $S_{n_{k_5}} \varepsilon (S_1, \cdots, S_a, \cdots, S_{m_{k_6}}, \cdots, S_{n_{k_5}-1})$, thus contradicting the hypothesis that $S_1, \cdots, S_n$ defines a composition series for $\mathfrak{G}$. Thus our assumption that the theorem is false cannot be true and the result is proved.

THEOREM D. *If in the set $S_1, \cdots, S_n$ we replace the element S_1 by an element T of the group $\mathfrak{G}$ so that the resulting set forms a coverage system for $\mathfrak{G}$, then $(S_1) = (T)$.*

PROOF. We first prove the lemma:

If $S_1, \cdots, S_n$ be any set defining a composition series for $\mathfrak{G}$, $S_2, \cdots, S_n$ cannot be a coverage system.

This is clear for cyclic groups. We prove it by induction, assuming it true for all groups with composition series of length less than $n+1$. Let us assume it false for $\mathfrak{G}$, that is, $S_2, \cdots, S_n$ is a coverage system. Then using (1) and applying Theorem B, we see $S_1, \cdots, S_{a-1}, S_{a+1}, \cdots, S_n$ defines a composition series for $\mathfrak{G}/(X)$ and $S_2, \cdots, S_{a-1}, S_{a+1}, \cdots, S_n$ defines a coverage system. As $\mathfrak{G}/(X)$ has composition series of length n this contradicts the induction assumption and so the lemma is proved.

The proof of the theorem itself is similar. It is clear for cyclic groups. We make the induction assumption that it is true for all groups with composition series of length less than $n+1$.

The set $X, T_1, \cdots, T_k$ defines a composition series for the group $\mathfrak{G}_X$; by (3) S_a is different from each of $T_1, \cdots, T_k$. With the help of these facts and (1), Theorem B may be easily generalized to show (1) that the restclasses defined by $S_1, \cdots, S_n$ after the omission of $S_a, T_1, \cdots, T_k$ define a composition series for the factor group $\mathfrak{G}/\mathfrak{G}_X$ and (2) that the restclasses defined by $T, S_2, \cdots, S_n$ after the omission of $S_a, T_1, \cdots, T_k$ define a coverage system for $\mathfrak{G}/\mathfrak{G}_X$. None of the restclasses remaining from the set $S_2, \cdots, S_n$ has order p

in $\mathfrak{G}/\mathfrak{G}_X$, for otherwise the set $X, T_1, \cdots, T_k$ would not be a greatest possible set. Then as $\mathfrak{G}/\mathfrak{G}_X$ has composition series of length less than $n+1$, we have by the induction assumption that T and S_1 generate the same group within $\mathfrak{G}/\mathfrak{G}_X$. Therefore $T=S_1^{x_1}U$, $(p\nmid x_1,\ U\in\mathfrak{G}_X)$.

Now let $Y=S_1X$. As $S_1\in(S_{a-1})$, we have from the definition of a $(S_{a-1}, S_a)=(S_{a-1}, X)=(S_{a-1}, Y)$. Hence Y satisfies the same conditions as X and we may therefore build a group $\mathfrak{G}_Y$ analogous to $\mathfrak{G}_X$. Exactly as before we have $T=S_1^{x_2}V$, $(p\nmid x_2,\ V\in\mathfrak{G}_Y)$. Equating the two expressions for T, $S_1^{x_2}V=S_1^{x_1}U$; whence $(XS_1)^{x_2-x_1}V=X^{x_2-x_1}U$, that is, $Y^{x_2-x_1}V=X^{x_2-x_1}U$. Now the right-hand member of this last expression is within $\mathfrak{G}_X$, while the left-hand member is within $\mathfrak{G}_Y$. But as $(X)\neq(Y)$ by Theorem C, $\mathfrak{G}_X\cap\mathfrak{G}_Y=(E)$ and consequently $UX^{x_2-x_1}=E$. Then $U=X^{x_1-x_2}$ and $T=X^{x_1-x_2}S_1^{x_1}$.

Let us assume $p\nmid x_1-x_2$. Accordingly we may obtain an expression for X in terms of T and S_1 which we substitute into (1). By using $S_1\in(S_r)$, we thus obtain $S_a=T^{x_3}S_r^{x_4}$ and, as the order of S_a is greater than p, $p\nmid x_4$. Then by Theorem B the restclasses containing $S_1, \cdots, S_{a-1}, S_{a+1}, \cdots, S_n$ define a composition series for $\mathfrak{G}/(T)$, while those containing $T, S_2, \cdots, S_{a-1}, S_{a+1}, \cdots, S_n$ define a coverage system. As T is within the unit restclass it is redundant, and we thus arrive at a contradiction to the lemma. Hence x_1-x_2 must be divisible by p and then, as $X^p=E$, we have $(T)=(S_1)$, which proves the theorem.

THEOREM 1. *$S_1, \cdots, S_n$ defines a composition series for the group $\mathfrak{G}$ while $S_1, \cdots, S_{k-1}, T, S_{k+1}, \cdots, S_n$ is a coverage system for $\mathfrak{G}$. Then $S_1, \cdots, S_{k-1}, T, S_{k+1}, \cdots, S_n$ after a possible rearrangement of order defines a composition series for $\mathfrak{G}$.*

PROOF. For cyclic groups the result is evident. We prove the general result by induction according to the length of the composition series, assuming it true for all groups with composition series of length less than $n+1$. If the system $S_1, \cdots, S_{k-1}, S_{k+1}, \cdots, S_n$ contain no element of order p, then, as $S_1^p=E$, $k=1$. Hence by the previous theorem $(T)=(S_1)$, from which the result follows immediately. Therefore let the system contain an element S_m where $S_m^p=E$. By Theorem B, $S_1, \cdots, S_{m-1}, S_{m+1}, \cdots, S_n$ defines a composition series for the factor group $\mathfrak{G}/(S_m)$ while $S_1, \cdots, S_{k-1}, T, S_{k+1}, \cdots, S_{m-1}, S_{m+1}, \cdots, S_n$ is a coverage system. But as $\mathfrak{G}/(S_m)$ has composition series of length n it follows from the induction assumption that the restclasses defined by this latter system after a possible rearrangement of order define a composition series for the group $\mathfrak{G}/(S_m)$, which may be written in the form

$$(\bar{S}_1, \cdots, \bar{S}_{n-1}, S_m)/(S_m),\ (\bar{S}_1, \cdots, \bar{S}_{n-2}, S_m)/(S_m), \cdots,$$
$$(\bar{S}_1, S_m)/(S_m),\ (S_m)/(S_m),$$

where the $\bar{S}_1, \cdots, \bar{S}_{n-1}$ are merely a rearrangement of the terms $S_1, \cdots, S_{k-1}, T, S_{k+1}, \cdots, S_{m-1}, S_{m+1}, \cdots, S_n$. As $S_m^p = E$, it follows from the second isomorphism theorem that

$$(\bar{S}_1, \cdots, \bar{S}_{n-1}, S_m),\ (\bar{S}_1, \cdots, \bar{S}_{n-2}, S_m), \cdots, (\bar{S}_1, S_m),\ (S_m),\ (E)$$

is a composition series for $\mathfrak{G}$, which establishes the result.

THEOREM E. *A coverage system of a group of rank* 2 *has either an element of order p or two elements S_1, S_2 for which $S_2 = S_1^x S$, ($S^p = E$, $p \nmid x$).*

PROOF. For a group $\mathfrak{G}$ of rank 2 let Y be any element of order p and $\mathfrak{A}$ be the set of reciprocally cyclic subgroups of $\mathfrak{G}$ which are cyclic and contain the element Y. From the definition of a coverage system we see that it must contain, for every reciprocally cyclic subgroup $\mathfrak{H}$, an element S for which $(\mathfrak{H}, S):\mathfrak{H} = p$. From the elements of the coverage system we may therefore choose a least possible subset $\mathfrak{B}$ so that for every group $\mathfrak{H}$ of $\mathfrak{A}$ an element S of $\mathfrak{B}$ exists with $(\mathfrak{H}, S):\mathfrak{H} = p$. Let S_1 be any element of $\mathfrak{B}$ of maximum order. If $S_1^p = E$, the theorem is true; let us therefore assume $S_1^p \neq E$. Now a group $\mathfrak{H}$ exists within $\mathfrak{A}$ with $(\mathfrak{H}, S_1):\mathfrak{H} = p$, for otherwise we could omit S_1 from the set $\mathfrak{B}$, contrary to the assumption that $\mathfrak{B}$ is a least possible set. Clearly $S_1^p \in \mathfrak{H}$; then, as $\mathfrak{H}$ is cyclic and $S_1^p \neq E$, $Y \in (S_1^p) \leqq \mathfrak{H}$. Let T be an element of greatest possible order with $S_1 \in (T)$. Now $\mathfrak{G}$ having rank 2, it can be deduced from the fundamental theorem on abelian groups that (T) is reciprocally cyclic. Therefore $(T) \in \mathfrak{A}$ and so an element S_2 of $\mathfrak{B}$ exists with $(T, S_2):(T) = p$. Now as the order of S_2 cannot exceed that of S_1, $S_2^p \in (S_1^p) \leqq (T)$. Hence $S_2 = S_1^x S$, where $S^p = E$, because $S_2 \notin (T)$, $S \notin (S_1)$ and so (S, S_1^p) has rank 2. First let us assume $p \mid x$. Hence (S_2, S_1^p) would have rank 2 as it would contain (S, S_1^p). Furthermore for any group $\mathfrak{F}$ of $\mathfrak{A}$ for which $(\mathfrak{F}, S_1):\mathfrak{F} = p$, we should have $S_2^p \in (S_1^p) \leqq \mathfrak{F}$ while $S_2 \notin \mathfrak{F}$ for otherwise $(S_2, S_1^p) \leqq \mathfrak{F}$ and the cyclic group $\mathfrak{F}$ would contain the noncyclic group (S_2, S_1^p), which is impossible. Therefore $(S_2, \mathfrak{F}):\mathfrak{F} = p$, and consequently we could omit the element S_1 from the set $\mathfrak{B}$, contrary to hypothesis. Thus the assumption $p \mid x$ leads to a contradiction and so x must be prime to p. The theorem is then established.

Theorem E is not true in general for groups of rank higher than 2.*

* The following is a Gegenbeispiel: $\mathfrak{G}$ is a group of type (4 4 4) generated by A, B and C. The set of elements $A, B, C, ABC^2, BCA^2, AC^3B^2, AB^3C$ is easily verified to be a coverage system for $\mathfrak{G}$, for which Theorem E does not hold.

THEOREM 2. *A coverage system $S_1, \cdots, S_n$ for a group $\mathfrak{G}$ of rank 2 with composition series of length $n+1$ defines a composition series for $\mathfrak{G}$ after a possible rearrangement of order.*

PROOF. For cyclic groups the result is evident. We make the assumption that it is true for all groups of rank 2 with composition series of length less than $n+1$, and then prove it for the group $\mathfrak{G}$.

By Theorem E we may select from the coverage system of $\mathfrak{G}$ either an element $S=S_m$ of order p or two elements S_m, S_k for which $S_m=S_k^x S$, $(p\nmid x, S^p=E)$. By Theorem B, the restclasses defined by $S_1, \cdots, S_{m-1}, S_{m+1}, \cdots, S_n$ build a coverage system for the factor group $\mathfrak{G}/(S)$. But this factor group has at most rank 2 and composition series of length n. Hence, by the induction assumption, the restclasses defined by $S_1, \cdots, S_{m-1}, S_{m+1}, \cdots, S$, after a possible rearrangement of order, define a composition series which may be written in the form

$$(\overline{S}_1, \cdots, \overline{S}_{n-1}, S)/(S), (\overline{S}_1, \cdots, \overline{S}_{n-2}, S)/(S), \cdots, (\overline{S}_1, S)/(S), (S)/(S),$$

$\overline{S}_1, \cdots, \overline{S}_{n-1}$ being merely another arrangement of the terms $S_1, \cdots, S_{m-1}, S_{m+1}, \cdots, S_n$. Then, using the second isomorphism theorem, we see that

$$(\overline{S}_1, \cdots, \overline{S}_{n-1}, S), (\overline{S}_1, \cdots, \overline{S}_{n-2}, S), \cdots, (\overline{S}_1, S), (S), (E)$$

is a composition series for the group $\mathfrak{G}$. But the coverage system $S_1, \cdots, S_n$ is obtained from the set $\overline{S}_1, \cdots, \overline{S}_{n-1}, S$ by replacing the element S by the element S_m. Then we have immediately from Theorem 1 that $S_1, \cdots, S_n$, after a possible rearrangement of order, defines a composition series for the group $\mathfrak{G}$. The result is then proved.

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