ON THE COMPUTATION OF THE SECOND DIFFERENCES OF THE Si(x), Ei(x), AND Ci(x) FUNCTIONS*

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In the course of the computation of the functions†

$$(1)\quad \operatorname{Si}(x) = \int_0^x \frac{\sin \alpha}{\alpha}\, d\alpha = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)(2k+1)!},$$

$$(2)\quad \operatorname{Ci}(x) = \int_{-\infty}^x \frac{\cos \alpha}{\alpha}\, d\alpha = \gamma + \frac{1}{4}\log_e(x^4) + \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{2k\cdot(2k)!},$$

$$(3)\quad \operatorname{Ei}(x) = \int_{-\infty}^x \frac{e^{\alpha}}{\alpha}\, d\alpha = \gamma + \frac{1}{4}\log_e(x^4) + \sum_{k=1}^{\infty} \frac{x^{k}}{k\cdot k!},$$

it was felt advisable to precompute the second differences for the above functions. These second differences are of use in the Everett interpolation formula and may also be used as a check of the accuracy of the computed value. The object of this paper is to describe the method which was developed for the independent evaluation of the above second differences.

Let $\phi(x)$ stand for any of the three functions under consideration. Consider the expression

$$(4)\quad \begin{aligned} R(x) = {} & [\phi(x+h) + \phi(x-h) - 2\phi(x)] \\ & - \frac{h}{2}[\phi'(x+h) - \phi'(x-h)], \end{aligned}$$

where the first expression in brackets is the second difference to be evaluated.

Substituting for $\phi(x+h)$, $\phi(x-h)$, $\phi'(x+h)$, and $\phi'(x-h)$ their Taylor expansions, we get

$$(5)\qquad R(x) = \frac{-h^4}{12}\phi^{(4)}(x) + \sum_{k=3}^{\infty}\left[\frac{2}{(2k)!} - \frac{1}{(2k-1)!}\right]\cdot h^{2k}\phi^{(2k)}(x),$$

whence

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$$(6)\qquad |R(x)| < \frac{h^4}{12}|\phi^{(4)}(x)| + \sum_{k=3}^{\infty} \frac{2k-2}{(2k)!} h^{2k}|\phi^{(2k)}(x)|,$$

and a fortiori

$$(6')\qquad |R(x)| < \frac{h^4}{12}\{\phi^{(4)}(x)\} + \sum_{k=3}^{\infty} \frac{h^{2k}}{(2k-1)!}\{\phi^{(2k)}(x)\},$$

where, in general, the expression $\{\phi^{(2k)}(x)\}$ is an upper bound of the modulus of the $2k$th derivative of $\phi(x)$. We shall have a similar inequality for each one of the functions under consideration.

We proceed to obtain upper bounds for $\phi^{(2k)}(x)$.

Case of the function $\mathrm{Ei}(x)$. In this case

$$(7)\qquad \frac{d^{2k}}{dx^{2k}}\mathrm{Ei}(x) = \frac{d^{2k-1}}{dx^{2k-1}}\left(\frac{e^x}{x}\right).$$

Since $1/x=\int_0^\infty e^{-xt}dt$ for $x>0$, we can write $e^x/x=\int_0^\infty e^{x(1-t)}dt$, and therefore

$$(8)\qquad \begin{aligned} \frac{d^{2k-1}}{dx^{2k-1}}\left(\frac{e^x}{x}\right) &= -e^x\int_0^\infty (t-1)^{2k-1}e^{-xt}dt \\ &= -e^x\int_0^1 (t-1)^{2k-1}e^{-xt}dt - e^x\int_1^\infty (t-1)^{2k-1}e^{-xt}dt. \end{aligned}$$

Consider the first integral of (7). Since for $x>0$ and $t>0$ we have $e^{-xt}<1$, it follows that

$$(9)\qquad \int_0^1 (t-1)^{2k-1}e^{-xt}dt < \int_0^1 (t-1)^{2k-1}dt = \frac{1}{2k}\cdot$$

Consider now the second integral of (7). If we make the substitution $t-1=\eta$, we get

$$(10)\qquad \int_1^\infty (t-1)^{2k-1}e^{-xt}dt = e^{-x}\int_0^\infty \eta^{2k-1}e^{-x\eta}d\eta = \frac{e^{-x}(2k-1)!}{x^{2k}}\cdot$$

In view of (9) and (10), (8) yields

$$(11)\qquad \left\{\frac{d^{2k}}{dx^{2k}}\mathrm{Ei}(x)\right\} = \frac{e^x}{2k} + \frac{(2k-1)!}{x^{2k}}\cdot$$

Thus, for the function $\mathrm{Ei}(x)$, (6′) becomes

$$(12)\qquad |R(x)| < \frac{1}{2}\left(\frac{h}{x}\right)^4 + \frac{e^x}{48}\cdot\sum_{k=3}^{\infty}\frac{h^{2k}}{(2k-1)!}\left(\frac{e^x}{2k} + \frac{(2k-1)!}{x^{2k}}\right).$$

The three functions under consideration are being computed in the range $0<x<2$, at intervals $h=10^{-4}$, by adding an appropriate number of terms in the expansions (1), (2), and (3), each term being computed to 12 decimals, and rounding the sum to nine places of decimals.

From (12) it can be easily shown that for $x>0.1$, and $h=10^{-4}$, the sum of the terms beyond the first is of an order of magnitude not exceeding 10^{-16} and can therefore not affect the twelfth place of Ei(x). Thus (6′) may be written

$$|R(x)| < \frac{1}{2}\left(\frac{h}{x}\right)^4 \quad \text{or} \quad |R(x)| < 5 \times 10^{-13}.$$

We therefore reach the conclusion that the second difference of the function Ei(x) must agree in its 12 places of decimals with the value of

$$\frac{h}{2}\left(\frac{e^{x+h}}{x+h} - \frac{e^{x-h}}{x-h}\right).$$

Case of the function Si(x). In this case we have

$$\frac{d^{2k}}{dx^{2k}}\,\mathrm{Si}(x) = \frac{d^{2k-1}}{dx^{2k-1}}\left(\frac{\sin x}{x}\right). \tag{13}$$

Since

$$\frac{\sin x}{x} = \int_0^1 \cos xt\,dt,$$

the preceding equation yields

$$\left|\frac{d^{2k}}{dx^{2k}}\,\mathrm{Si}(x)\right| < \int_0^1 t^{2k-1}|\sin xt|\,dt, \tag{14}$$

or

$$\left|\frac{d^{2k}}{dx^{2k}}\,\mathrm{Si}(x)\right| < \int_0^1 t^{2k-1}dt = \frac{1}{2k}\cdot \tag{14′}$$

Thus, in the case of the Si(x) function, the inequality (6′) becomes

$$|R(x)| < \frac{h^4}{48} + \sum_{k=3}^{\infty}\frac{h^{2k}}{(2k)!}\cdot \tag{15}$$

For $h=10^{-4}$ the second member of (15) is of the order of 10^{-17}. We therefore reach the conclusion that, for the entire range of x under consideration, the second difference of the function Si(x) must agree in its twelve decimal places with the values of

$$\frac{h}{2}\left[\frac{\sin (x+h)}{x+h}-\frac{\sin (x-h)}{x-h}\right].$$

Case of the function Ci(x). In this case we have

$$\frac{d^{2k}}{dx^{2k}}\,\mathrm{Ci}(x)=\frac{d^{2k-1}}{dx^{2k-1}}\left(\frac{\cos x}{x}\right). \tag{16}$$

Since

$$\frac{\cos x}{dx}=\int_0^\infty e^{-xt}dt-\int_0^1 \sin xt\,dt,$$

we ultimately get

$$\left|\frac{d^{2k}}{dx^{2k}}\,\mathrm{Ci}(x)\right|<\int_0^\infty t^{2k-1}e^{-xt}dt+\int_0^1 t^{2k-1}dt=\frac{(2k-1)!}{x^{2k}}+\frac{1}{2k}. \tag{17}$$

Thus, in the case of the Ci(x) function, the inequality (6′) becomes

$$|R(x)|<\frac{1}{2}\left(\frac{h}{x}\right)^4+\frac{h^4}{48}+\sum_{k=3}^{\infty}\left(\frac{1}{x^{2k}}+\frac{1}{(2k)!}\right)h^{2k}. \tag{18}$$

As in the case of the Ei(x) function (and for similar reasons), we reach the concludion that for $x>0.1$ and $h=10^{-4}$, the second difference of the function Ci(x) must agree in its first 12 places of decimals with the values of

$$\frac{h}{2}\left[\frac{\cos (x+h)}{x+h}-\frac{\cos (x-h)}{x-h}\right].$$

It is conceivable that if the second differences of the functions under consideration are computed for arguments separated by some suitable interval H, the second differences for some intermediate arguments will be obtainable by linear interpolation to a high degree of accuracy.

Let $\Delta_2\phi(x)$ designate the second difference of any of the functions under consideration and $E(x)$ the error in the value obtained by linear interpolation between the arguments x and $x+h$. Then

$$E(x+pH)=(1-p)\Delta_2\phi(x)+p\Delta_2\phi(x+H)-\Delta_2\phi(x+pH), \tag{19}$$

where $0<p<1$.

Replacing the 2nd and 3rd terms by their Taylor expansions, we get

$$
\begin{aligned}
E(x + pH) &= \frac{H^2}{2}(p - p^2)\Delta_2''(x) + \frac{H^3}{3!}(p - p^3)\Delta_2^{(3)}(x) + \cdots \\
&\quad + \frac{H^n}{n!}(p - p^n)\Delta_2^{(n)}(x) + \cdots .
\end{aligned} \tag{20}
$$

But

$$
\begin{aligned}
\Delta_2\phi(x) &= \phi(x - h) + \phi(x + h) - 2\phi(x) \\
&= h^2\phi''(x) + \frac{h^4}{12}\phi^{(4)}(x) + \cdots , \qquad h = 10^{-4}.
\end{aligned} \tag{21}
$$

Substituting (21) in (20), we get

$$
E(x + pH) = \frac{H^2h^2}{2}(p - p^2)\phi^{(4)}(x) + \frac{H^3h^2}{3!}\phi^{(5)}(x) + \cdots .
$$

If we substitute for $(p-p^2)$ the maximum value $\frac{1}{4}$, and replace $(p-p^3)$, $(p-p^4)$, $\cdots$ by their upper bound 1, we get

$$
| E(x + pH) | < \frac{10^{-8}H^2}{8}\{\phi^{(4)}(x)\} + \frac{10^{-8}H^3}{6}\{\phi^{(5)}(x)\} + \cdots . \tag{22}
$$

If we assume the condition $|E(x+pH)| < 10^{-11}/2$, and arbitrarily set $H = 10^{-2}$, the above inequality will yield lower limits of the argument x, above which the second differences may be computed at intervals of 10^{-2}, the second differences for intermediate arguments being obtained by linear interpolation accurately to within $10^{-11}/2$. Thus, in the case of the function Ei(x), the evaluation of the second member of (22) leads to the conclusion that if $x > 0.7$, then $|E(x+pH)| < 3 \times 10^{-12}$. It thus suffices to compute the second differences at intervals of 0.0100, in the manner described in the first part of this paper.

Similarly, if we put $H = 10^{-2}/2$, the evaluation of the second member of (22) leads to the result that if $x > 0.5$, then $|E(x+pH)| < 3 \times 10^{-12}$.

Thus, in the interval $0.5 < x < 0.7$, it suffices to compute the second differences at intervals of 0.0050.

Entirely similar results are obtained in the case of the Ci(x) function. Finally, in the case of the Si(x) function, we reach the conclusion that over the entire range of the argument x, it suffices to compute the second differences at intervals 0.0100.

The method here described is based on the suggestions made by Mr. Frederick King. These suggestions have led to the evaluation of $R(x)$ as a starting point of the subsequent discussion.

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ALL INTEGERS EXCEPT 23 AND 239 ARE SUMS OF EIGHT CUBES

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Summary. In 1770 Waring stated that every positive integer is a sum of nine integral nonnegative cubes. The first proof is due to Wieferich.* I shall prove the following new result.

Theorem. *Every positive integer other than* 23 *and* 239 *is a sum of eight integral nonnegative cubes.*

Five lemmas are required.

Lemma 1. *Every integer greater than or equal to* $233^6 D$ *is a sum of eight cubes if* $D = 14.0029682$, *or more generally if* $D = d$, *where*†

$$d > 14 + \left(\frac{24}{167}\right)^3, \qquad d \leqq 14.1.$$

The algebraic part of Wieferich's proof holds for all integers exceeding $2\frac{1}{4}$ billion. The fact that all smaller integers are sums of nine cubes was proved by use of Table I. To prove my theorem, I shall need also the new Tables II and III.

Table I gives, for each positive integer $N \leqq 40{,}000$, the least number m such that N is a sum of m cubes.

It was computed by R. D. von Sterneck‡ by adding all cubes to

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* His errors are avoided in the much simpler proof by the writer, Transactions of this Society, vol. 30 (1928), pp. 1–18. On page 16 is proved a generalization of Landau's result that all sufficiently large numbers are sums of eight cubes.

† The proof is essentially like that given for $d = 14.1$ by W. S. Baer, *Beiträge zum Waringschen Problem*, Dissertation, Göttingen, 1913.

‡ Sitzungsberichte der Akademie der Wissenschaften, Vienna, IIa, vol. 112 (1903), pp. 1627–1666.