SPACES OF UNCOUNTABLY MANY DIMENSIONS*

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Riemann in his *Habilitations Schrift* of 1854 suggested the notion of n-dimensional space (where n is a natural number) as an extension of the notion of three-dimensional euclidean space. Hilbert extended the notion still further by defining a space of a countably infinite number of dimensions. Fréchet† in 1908 defined two other spaces of countably many dimensions, which he called D_ω and E_ω. Tychonoff‡ in 1930 defined a series of spaces of an unlimited number of dimensions and established several of their properties. The present paper undertakes, by generalizing the notions of spaces D_ω and E_ω, to define spaces D^α and E^α, respectively, for each cardinal number $\aleph_\alpha$ representing the number of dimensions. It is shown that every metric space is homeomorphic with a subset of some space D^α. Certain properties of Tychonoff's spaces, here called spaces T^α, are also presented.

1. **Spaces** D^α. For each initial ordinal number ω_α the set of all points of space D^α is the set of all type ω_α sequences $[x_i]_i^{\omega_\alpha}$ of real numbers x_i, such that $0<x_i<1$. Suppose that $[P_i]_i^{\omega_0}$ is a type ω_0 sequence of points of space D^α such that for each i, $P_i=[x_{i,j}]_j^{\omega_\alpha}$; and suppose that $P=[y_j]_j^{\omega_\alpha}$ is a point of space D^α. The sequence $[P_i]_i^{\omega_0}$ is said to *converge* to the point P if and only if it is true that if ϵ is a positive number, there exists a positive integer N_ϵ such that if $i>N_\epsilon$, then $|y_j-x_{i,j}|<\epsilon$ for every value of $j<\omega_\alpha$. The point P is said to be the *sequential limit point* of $[P_i]_i^{\omega_0}$. A point P is said to be a *limit point* of a point set M, provided there exists a sequence of distinct points of M which converges to P. Thus space D^0 is equivalent to space D_ω of Fréchet. For each two distinct points $A=[x_i]_i^{\omega_\alpha}$ and $B=[z_i]_i^{\omega_\alpha}$ of D^α such that for each i, $x_i\neq z_i$, let $D^\alpha_{(A,B)}$, also referred to as *segment* AB, denote the set of all points $P=[y_i]_i^{\omega_\alpha}$ such that $x_i<y_i<z_i$ or $z_i<y_i<x_i$. If there exist constants h and k, $(h<k)$, such that for each i, $x_i=h$ and $y_i=k$, the notation $D^\alpha_{(h,k)}$ is used. It is evident that $D^\alpha_{(h,k)}$ is homeomorphic with D^α for every value of α, h, and k; but there exist points A and B such that $D^\alpha_{(A,B)}$ is not homeomorphic with D^α.

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† M. Fréchet, *Essai de géométrie analytique à une infinité de coordinées*, Nouvelles Annales de Mathématique, (4), vol. 8 (1908), pp. 97–116. Also *Les Espaces Abstraits*, Paris, 1928, pp. 81–84, 97–99.

‡ Mathematische Annalen, vol. 102 (1930), pp. 544–561.

THEOREM 1.1. *Space D^α is metric, for every value of α.*

Suppose that $A=[x_i]_i^{\omega_\alpha}$ and $B=[y_i]_i^{\omega_\alpha}$ are points of D^α. Let the distance from A to B, $d(A, B)$, be the least upper bound of the numbers of the sequence $[|y_i-x_i|]_i^{\omega_\alpha}$. This definition can be shown to satisfy the conditions of Fréchet for metric spaces, as in the case of space D_ω.

THEOREM 1.2. *In space D^α there exists a point set of power $2^{\aleph_\alpha}$ which has no limit point.*

Let M denote the set of all points P such that each of the coordinates of P is either 1/4 or 1/2. Then M is of power $2^{\aleph_\alpha}$. If A and B are distinct points of M, then $d(A, B)=1/4$. Hence no point of M is a limit point of M. But M is closed since any limit point of M has each of its coordinates either 1/4 or 1/2. Hence M has no limit point.

THEOREM 1.3. *Space D^α is not $\aleph_x$-separable* if $\aleph_x<2^{\aleph_\alpha}$.*

Suppose the theorem is false. There exists a cardinal number $\aleph_\beta<2^{\aleph_\alpha}$ such that there exists a subset K of power $\aleph_\beta$, everywhere dense in D^α. Hence D^α is $\aleph_\beta$-completely separable.* Let G denote a collection of power $\aleph_\beta$ of domains with respect to which D^α is $\aleph_\beta$-completely separable. By Theorem 1.2 there exists a point set of power $2^{\aleph_\alpha}$ which has no limit point. For every point P of M there exists a domain of G which contains P and no other point of M; and this involves a contradiction.

THEOREM 1.4. *If M is an $\aleph_\alpha$-separable metric space, then M is homeomorphic with a subset of space D^α.*

This theorem is a generalization of a theorem of Urysohn,† and the proof is a generalization of that of Urysohn.

There exists a distance function $d(A, B)$ such that if A and B are distinct points of M, then $0<d(A, B)<1$. Let K denote a subset of M, of power $\aleph_\alpha$, everywhere dense in M. Let $[K_i]_i^{\omega_\alpha}$ denote a type ω_α sequence of all the elements of K. For each point P of M let

$$\phi(P) = [1/2 + 1/2d(P, K_i)]_i^{\omega_\alpha}.$$

The correspondence is one-to-one. To show this, it is sufficient to

* For a definition of this term see C. W. Vickery, *Spaces in which there exist uncountable convergent sequences of points*, Tôhoku Mathematical Journal, vol. 40 (1934), p. 11. Cf. Keitaro Haratomi, *Über höherstufige Separabilität und Kompacktheit*, Japanese Journal of Mathematics, vol. 8 (1931), pp. 113–141; vol. 9 (1932), pp. 1–18.

† P. Urysohn, *Les classes (D) separables et l'espace Hilbertien*, Comptes Rendus de l'Académie des Sciences, vol. 178 (1924), p. 65.

show that if A and B are distinct points, $\phi(A) \neq \phi(B)$. Suppose that A and B are distinct points. There exists a point K_j of K such that

$$d(A, K_j) < 1/3d(A, B).$$

But

$$d(A, K_j) + d(K_j, B) \geqq d(A, B).$$

Hence $d(A, K_j) < d(K_j, B)$, and hence $\phi(A) \neq \phi(B)$.

The correspondence is also bicontinuous. Suppose that ϵ is a positive number and P and Q are points of M such that $d(P, Q) < \epsilon$. Let $\phi(P) = [x_i]_i^{\omega_\alpha}$ and $\phi(Q) = [y_i]_i^{\omega_\alpha}$. Suppose further that K_i is an element of $[K_i]_i^{\omega_\alpha}$. Then

$$d(P, K_i) + d(P, Q) \geqq d(Q, K_i), \quad d(Q, K_i) + d(P, Q) \geqq d(P, K_i).$$

Since $d(P, Q) < \epsilon$, it follows that

$$\left| d(Q, K_i) - d(P, K_i) \right| < \epsilon$$

and hence

$$\left| x_i - y_i \right| < \epsilon.$$

Since this relation holds for every value of i, it follows that if P is a limit point of a point set H, then $\phi(P)$ is a limit point of $\phi(H)$.

Suppose now that δ is a positive number and P and Q are points of M such that $d(P, Q) > \delta$. Let $\phi(P) = [x_i]_i^{\omega_\alpha}$, $\phi(Q) = [y_i]_i^{\omega_\alpha}$, and $d(P, Q) = \delta + \eta$. There exists an ordinal number x such that $d(Q, K_x) < \eta/2$. It follows that

$$\left| d(P, K_x) - d(P, Q) \right| < \eta/2.$$

Hence

$$d(P, K_x) > \delta + \eta/2,$$

hence

$$d(P, K_x) - d(Q, K_x) > \delta,$$

and hence

$$\left| x_x - y_x \right| > \delta/2.$$

Therefore if P is not a limit point of a point set H, then $\phi(P)$ is not a limit point of $\phi(H)$.

Note. Since space D^α is not $\aleph_x$-separable for any cardinal number $\aleph_x < 2^{\aleph_\alpha}$, the power of the space, it thus appears that there is a relation between the dimensionality of a metric space and its type of separability. The type of separability does not determine the dimensionality, but it fixes an upper bound to it.

THEOREM 1.5. *In order that a space S be metric, it is necessary and sufficient that there exist a cardinal number $\aleph_\alpha$ such that S is homeomorphic with a subset of space D^α.*

This follows with the help of Theorems 1.3, 1.4, and the fact that every metric space is $\aleph_x$-separable for some cardinal number $\aleph_x$.

DEFINITION. *A collection G of real-valued functions defined over a point set M is said to be equi-continuous provided it is true that if P is a point of M and ϵ is a positive number, there exists an open subset D of M containing P such that if f is a function of G, the oscillation of f on D is less than ϵ.*

THEOREM 1.6. *In order that a Hausdorff topological space S be homeomorphic with a subset of space D^α, it is necessary and sufficient that there exist an equi-continuous collection G, of power $\aleph_\alpha$, of functions defined over S, whose values are real numbers between 0 and 1, such that if P is a point and H is a closed point set not containing P, there exists a positive number δ such that if Q is a point of H, there exists a function f of G such that*

$$|f(Q) - f(P)| > \delta.$$

The condition is sufficient. Let $\gamma = [f_i]_i^{\omega_\alpha}$ denote a type ω_α sequence of the elements of G. For each point P, let $\phi(P) = [f_i(P)]_i^{\omega_\alpha}$. Then ϕ represents a one-to-one continuous transformation of S into a subset of space D^α.

The condition is also necessary. Suppose that $\phi(P)$ is a one-to-one continuous transformation of S into a subset of space D^α. For each ordinal number $\nu < \omega_\alpha$ and each point P, where $\phi(P) = [x_i]_i^{\omega_\alpha}$, let $f_\nu(P) = x_\nu$. Let G denote the collection of all functions f_ν so defined. Then G is the required collection.

2. **Spaces E^α.** If the conditions, employed in defining spaces D^α, for the convergence of a sequence of points are made less restrictive so that it is no longer required that the sequences of coordinates converge *uniformly*, spaces E^α are defined. Using the notation of §1, the sequence of points $[P_i]_i^{\omega_0}$ is said to *converge* to the point P if and only if it is true that if j is an ordinal number less than ω_α and ϵ is a positive number, then there exists a positive integer $N_{\epsilon,j}$ such that if $i > N_{\epsilon,j}$ then $|y_j - x_{i,j}| < \epsilon$. Space E^0 is thus equivalent to space E_ω of Fréchet. Every segment $E^\alpha_{(A,B)}$ is homeomorphic with E^α.

THEOREM 2.1. *There exists in space E^1 a compact, non-separable point set.*

Let H denote the set of all points $P=[x_j]_j^{\omega_1}$ such that for each j, $1/4\leqq x_j\leqq 1/2$ and such that there exists an ordinal number $\delta_P<\omega_1$ such that if $j>\delta_P$, then $x_j=1/4$. This set is compact. Suppose that $[P_i]_i^{\omega_0}$ is a type ω_0 sequence of points $P_i=[x_{i,j}]_j^{\omega_1}$ of H; this sequence has a convergent subsequence. For each point P_i there exists an ordinal number δ_i such that if $j>\delta_i$ then $x_{i,j}=1/4$. The sequence of elements δ_i is a type ω_0 sequence; let μ denote the smallest ordinal number greater than all the elements of this sequence. Then $\mu<\omega_1$. Let f denote a one-to-one correspondence between the ordinal numbers less than ω_0 and the ordinal numbers less than μ. Let K denote the set of all points $Q=[y_i]_i^{\omega_1}$ such that if $i>\mu$ then $y_i=1/4$. For each point $Q=[y_i]_i^{\omega_1}$ of K let $\phi(Q)=[y_j]_{f(j)}^{\omega_0}$. Then ϕ represents a one-to-one bicontinuous transformation of the points of K into a subset of space E^0 such that if z_h, $(h<\omega_0)$, is a coordinate of a point of $\phi(H)$, then $1/4\leqq z_h\leqq 1/2$. But in space E^0 the set M of all points whose coordinates are greater than or equal to $1/4$ and less than or equal to $1/2$ is closed and compact. There exists a subsequence of $[\phi(P_i)]_i^{\omega_0}$, namely $[\phi(P_{i_n})]_{i_n}^{\omega_0}$, which converges to a point T of M. Hence $[P_{i_n}]_{i_n}^{\omega_0}$ converges to $\phi^{-1}(T)$.

The point set H is not separable. Suppose that it is separable. Then there exists a countable everywhere dense subset M. There exists an ordinal number $\xi<\omega_1$ such that if $P=[x_i]_i^{\omega_1}$ is a point of M, then if $i>\xi$, $x_i=1/4$. But no point which fails to satisfy this condition is a limit point of M, and since there are uncountably many such points, this leads to a contradiction.

THEOREM 2.2. *Space E^α is not metric for $\alpha\geqq 1$.*

By Theorem 2.1, E^1 contains a compact, non-separable subset H. Hence if $\alpha\geqq 1$, E^α contains a compact, non-separable subset. But every compact subset of a metric space is separable. Hence E^α is not metric for $\alpha\geqq 1$.

THEOREM 2.3. *If S is a normal, $\aleph_\alpha$-completely separable, topological space (of Hausdorff), then S is homeomorphic with a subset of the space E^α.*

This theorem is a generalization of a theorem of Urysohn* to the effect that every normal, completely separable, Hausdorff topological

* See W. Sierpiński, *Introduction to General Topology*, translated by C. C. Krieger, Toronto, 1934, p. 92.

space is homeomorphic with a subset of Hilbert space. Tychonoff has established a similar theorem for spaces T^α.

THEOREM 2.4. *If S is a normal topological space (of Hausdorff), there exists a cardinal number $\aleph_\alpha$ such that S is homeomorphic with a subset of space E^α.*

This follows with the help of Theorem 2.3 and the fact that S is $\aleph_\alpha$-completely separable for some value of $\aleph_\alpha$.

3. **Spaces T^α.** By a further extension of the notion of limit point, Tychonoff's spaces, here called spaces T^α, are produced. In order to define limit point in these spaces, the notion of region is introduced. Suppose that G_α denotes the collection of all finite sets of ordinal numbers $\nu<\omega_\alpha$. Then G_α is of power $\aleph_\alpha$. Let $H_\alpha=[h_i]_i^{\omega_\alpha}$ denote a type ω_α sequence of all the elements of G_α. If ν is an ordinal number less than ω_α, r a positive real number, and $A=[a_i]_i^{\omega_\alpha}$ a point, let $R_{(A,\nu,r)}$ denote the set of all points $P=[x_i]_i^{\omega_\alpha}$ such that if i is an element of h_ν, then $(a_i-r)<x_i<(a_i+r)$. Then $R_{(A,\nu,r)}$ is called a *region.* The region $R_{(A,\nu,r)}$ is said to have *center* at A, *constrictions* at h_ν, and *radius* r. For all values of A, ν, and r, $R_{(A,\nu,r)}$ is an open set in space E^α. If we now adopt the definition that a point P is a *limit point* of a point set M if and only if every region that contains P contains a point of M distinct from P, we find that there exist point sets M and points P such that P is a limit point of M in this sense, but not in the sense of spaces E^α. This new definition thus constitutes an extension of the notion of limit point and the resulting spaces are designated by T^α. For example, consider the point set M in space T^1 consisting of all points $Q=[x_{i,j}]_j^{\omega_1}$, where $x_{i,j}=1/4$ for $j<i$ and $x_{i,j}=1/2$ for $j\geqq i$. The point set M does not have the point $P=[y_j]_j^{\omega_1}$ (where for each j, $y_j=1/4$) as a limit point in space E^1. However, P is a limit point of M in space T^1.

THEOREM 3.1. *For each cardinal number $\aleph_\alpha$, space T^α is a distributive space $\Gamma_{\aleph_\alpha}$.*

A space is said to be a space $\Gamma_{\aleph_\alpha}$ provided it satisfies Axiom $1_{(\aleph_\alpha)}$, and a distributive space provided it satisfies Axiom 2 of the author's paper *Spaces in which there exist uncountable convergent sequences of points.**

It will be shown that there exists a family F of power $\aleph_\alpha$ satisfying the conditions of Axiom $1_{(\aleph_\alpha)}$. For each ordinal number $\nu<\omega_\alpha$ and each natural number n, let $G_{(\nu,n)}$ denote the collection of all regions of

* C. W. Vickery, op. cit., pp. 12–13.

radius $1/n$ having restrictions at h_ν. Let F denote the family of all collections $G_{(\nu,n)}$. It can easily be shown that F is the required family. Axiom 2 is evidently satisfied.

Note. If in space T^α the point P be said to be the *sequential limit point* of a type ω_α sequence of points $[P_i]_i^{\omega_\alpha}$ if and only if it is true that if R is a region containing P, then R contains a residue of sequence $[P_i]_i^{\omega_\alpha}$, there then exist type ω_α convergent sequences of points. Thus we have been led by a series of apparently natural definitions to the existence of uncountable convergent sequences of points in certain spaces of uncountably many dimensions.

AUSTIN, TEX.

NOTE ON THE LOCATION OF ZEROS OF THE DERIVATIVE OF A RATIONAL FUNCTION WHOSE ZEROS AND POLES ARE SYMMETRIC IN A CIRCLE*

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1. **Introduction.** The most general function which effects a 1-to-m conformal transformation of the interior of the unit circle $|z| = 1$ onto itself is of the form

$$r(z) = \lambda \prod_{k=1}^{m} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z}, \quad |\alpha_k| < 1, \quad |\lambda| = 1; \tag{1}$$

so the location of the zeros of the derivative $r'(z)$ is of considerable interest. The zeros and poles of $r(z)$ are symmetric in the unit circle. Moreover a typical transcendental function bounded in the unit circle is the Blaschke product (assumed convergent)

$$B(z) = \prod_{k=1}^{\infty} \frac{\bar{\alpha}_k}{|\alpha_k|} \frac{z - \alpha_k}{\bar{\alpha}_k z - 1}, \tag{2}$$

which is the limit for $|z| < 1$ of a sequence of functions each of form (1). It is of some significance in studying the behavior of $B(z)$ to know exactly or approximately the zeros of $B'(z)$. The object of the present note is to give some fairly simple but elegant results on the derivatives of both $r(z)$ and $B(z)$. Application is made also to the critical points of certain harmonic functions.

2. **Derivative of a rational function.** We first obtain the following result:

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