$$K(x, y) = \begin{cases} f_s(y) \text{ if } x^1 = s < 1 \text{ and } \int_0^\eta f_s < \infty, \\ \prod_{j=1}^m 1/x^j \text{ if } x^1 \geqq 1 \text{ and } 0 < y^j < x^j \text{ for all } j, \\ 0 \text{ otherwise.} \end{cases}$$

Then K is regular on $\mathfrak{B}_m$; so $\mathfrak{R} \supset \mathfrak{B}_m$. If $f \ \varepsilon \ \mathfrak{W}_m - \mathfrak{B}_m$, either there is a $b > 0$ such that B is of finite measure and $\int_B f$ exists or not; if there is no such b, $|f|$ dominates a function with this property. Hence there is a function $f_0(y)$, zero if y is not in B, such that $\int_B f_0$ exists and $\int_B f_0 f$ does not. Then there is an $s < 1$ such that f_s dominates f_0 but $\int_0^\eta f_s < \infty$. Hence there is an x such that $U_x(f)$ does not exist; so $\mathfrak{B}_m \supset \mathfrak{R}$.

For $m \geqq 2$ this construction can be modified to give a kernel with $\mathfrak{R}_m$ as domain of regularity.

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BOUNDED SELF-ADJOINT OPERATORS AND THE PROBLEM OF MOMENTS*

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It is known that there exists a close connection between the theory of moments and Jacobi matrices on one side, and the theory of self-adjoint operators in Hilbert space on the other. This connection has been thoroughly investigated by Stone in the tenth chapter of his textbook on Hilbert space.† The solution of both the moment problem and the spectral resolution of self-adjoint operators relies on the possibility of representing a class of analytic functions with positive imaginary parts in the upper half-plane by *Stieltjes* integrals of the form

$$\int_{-\infty}^{+\infty} \frac{d\rho(\lambda)}{\lambda - z} .$$

However, the spectral resolution requires the representation of more general functions than those involved in the solution of the problem of moments. The *bounded* self-adjoint operators do not, and I wish to show that the spectral theorem for *bounded* operators can be deduced immediately from the well known theorems concerning the problem of moments. Let H be a bounded self-adjoint operator, f an element of the Hilbert space, $R_z = (H - zI)^{-1}$ the resolvent of H. It

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is known that in order to prove the standard form of the spectral theorem it is sufficient to prove that given any f there exists a unique monotone increasing function $\rho(\lambda)$ of bounded variation, such that

$$(R_z f, f) = \int_{-\infty}^{+\infty} \frac{d\rho(\lambda)}{\lambda - z}$$

for all values of z.* The resolvent of a *bounded* self-adjoint operator can be developed in a series:

$$- R_z = \frac{I}{z} + \frac{H}{z^2} + \frac{H^2}{z^3} + \cdots,$$

or

$$- (R_z f, f) = \frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \cdots,$$

where $c_i = (H^i f, f)$, and the series is convergent for sufficiently large values of z.

Now, if the moment problem has a solution for the sequence $\{c_i\}$; that is, if there exists a $\rho(\lambda)$ such that

$$c_i = \int_{-\infty}^{+\infty} \lambda^i d\rho(\lambda)$$

for all i, then

$$(R_z f, f) = - \sum_{i=0}^{\infty} \int_{-\infty}^{+\infty} \frac{\lambda^i}{z^{i+1}} d\rho(\lambda) = \int_{-\infty}^{+\infty} \frac{d\rho(\lambda)}{\lambda - z},$$

and the spectral problem is solved.

Let us consider the quadratic forms

$$\left| \sum_{i=0}^{n} x_i H^i f \right|^2 = \sum_{i,j=0}^{n} x_i \bar{x}_j (H^i f, H^j f) = \sum_{i,j=0}^{n} c_{i+j} x_i \bar{x}_j, \qquad n = 1, 2, \cdots.$$

These forms are all definite positive for every f and n; therefore their determinants

$$C_n = \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \cdot & \cdot & \cdots & \cdot \\ c_n & c_{n+1} & \cdots & c_{2n} \end{vmatrix}$$

* Cf. B. O. Koopman and J. L. Doob, this Bulletin, vol. 40 (1934), pp. 601–605, or B. A. Lengyel. Acta Scientiarum Mathematicarum, Szeged, vol. 9 (1939).

are all positive. This is the well known condition for the existence of a solution of the problem of moments.

In order to prove the uniqueness of the solution we make use of a theorem of Hamburger.* Let M_n be the minimum of the definite quadratic form

$$\sum_{i,j=0}^{n} c_{i+j} x_i \bar{x}_j$$

for $x_0 = 1$, and let N_n be the minimum of

$$\sum_{i,j=0}^{n} c_{i+j+2} x_i \bar{x}_j$$

for $x_0 = 1$. Evidently both M_n and N_n are nonnegative; furthermore $M_n \geq M_{n+1}$, $N_n \geq N_{n+1}$; therefore, $M = \lim M_n$ and $N = \lim N_n$ exist.

The necessary and sufficient condition for the uniqueness of the solution of the moment problem is that at least one of the numbers M and N is 0.

Thus, to prove the uniqueness of $\rho(\lambda)$ we have to show that for every f, at least one of the sequences

$$M_n = \min \left| f - \sum_{i=1}^{n} x_i H^i f \right|^2, \qquad N_n = \min \left| Hf - \sum_{i=1}^{n} x_i H^{i+1} f \right|^2$$

tends to 0. This means that f can be approximated by linear combinations of $Hf, H^2f, \cdots$; or Hf can be approximated by linear combinations of $H^2f, H^3f, \cdots$; or both approximations are possible. In other words, if $\mathfrak{M}_f$ is the closed linear manifold determined by $Hf, H^2f, \cdots$, then at least one of the following inclusion relations holds:

(1) $$f \, \varepsilon \, \mathfrak{M}_f,$$

(2) $$Hf \, \varepsilon \, \mathfrak{M}_{Hf}.$$

This lemma is interesting in itself; its proof can be carried out as follows.

Assume that $\mathfrak{M}_n$ is the closed linear manifold determined by $[Hf, H^2f, \cdots, H^n f]$. Evidently $\mathfrak{M}_n \subset \mathfrak{M}_{n+1}$ and $H\mathfrak{M}_n \subset \mathfrak{M}_{n+1}$. Therefore, $\mathfrak{M}_f = \lim_{n \to \infty} \mathfrak{M}_n$ exists and $H\mathfrak{M}_f \subset \mathfrak{M}_f$. The element f can be re-

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* H. Hamburger, Mathematische Annalen, vol. 81 (1920), p. 234. Hamburger uses the same quadratic forms with real variables. This evidently does not affect the minima.

solved into the sum of f_1 and f_2, with f_1 in $\mathfrak{M}_f$ and f_2 in $\mathfrak{H} - \mathfrak{M}_f$, the orthogonal complement of $\mathfrak{M}_f$.

Either $f_2 = 0$, whence $f \varepsilon \mathfrak{M}_f$, or $f_2 \neq 0$. In the latter case we apply H^2 to $f = f_1 + f_2$ and multiply by f_2. This gives

$$(H^2 f, f_2) = (H^2 f_1, f_2) + (H^2 f_2, f_2).$$

Since $H^2 f$ and $H^2 f_1$ are in $\mathfrak{M}_f$, their scalar products with f_2 give 0; thus

$$(3) \qquad\qquad (H^2 f_2, f_2) = |Hf_2|^2 = 0.$$

The element f_1 is in the closed linear manifold $\mathfrak{M}_f$, so that for suitable $x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)}$ we have

$$\left| f - f_2 - \sum_{i=1}^{n} x_i^{(n)} H^i f \right| = \left| f_1 - \sum_{i=1}^{n} x_i^{(n)} H^i f \right| = \epsilon_n,$$

where $\epsilon_n \to 0$. Applying the bounded operator H to

$$f - f_2 - \sum_{i=1}^{n} x_i^{(n)} H^i f$$

and using (3), we obtain

$$\left| Hf - \sum_{i=1}^{n} x_i^{(n)} H^{i+1} f \right| \to 0, \quad \text{if} \quad n \to \infty;$$

that is, (2) holds. Thereby the proof of our statement is complete.

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