A CREMONA INVOLUTION IN S_3 WITHOUT A SURFACE OF INVARIANT POINTS*

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1. **Definition.** A Cremona involution in S_3, which is related to the involutions defined in a previous paper,† is of particular interest as it has no surface of invariant points.

Let

$$|H| \equiv H_1(x) - \lambda H_2(x) = 0 \tag{1}$$

be a pencil of quadric surfaces in S_3 and

$$\sum a_i x_i \equiv (ax) = 0, \qquad (bx) = 0 \tag{2}$$

a system of lines g in which each a_i is a polynomial of order m_1 in λ and each b_i a polynomial of order m_2 in λ.

The line g', polar conjugate of g as to H, is defined by

$$\begin{aligned} &r_{32}\frac{\partial H}{\partial x_1} + r_{13}\frac{\partial H}{\partial x_2} + r_{21}\frac{\partial H}{\partial x_3} = 0, \\ &r_{42}\frac{\partial H}{\partial x_1} + r_{14}\frac{\partial H}{\partial x_2} + r_{21}\frac{\partial H}{\partial x_4} = 0, \qquad r_{ik} = a_i b_k - a_k b_i. \end{aligned} \tag{3}$$

If y is any point in space, the H of the pencil through y is determined by $\lambda = H_1(y)/H_2(y)$. Through the point y passes one transversal line t of g and g'. Its equations are $(ax)(by)-(bx)(ay)=0=\sum r_{ik}p_{ik}=0$, where

$$p_{ik} = x_i y_k - x_k y_i, \tag{4}$$

and the corresponding equation from (3), which, after reduction by means of the quadratic identities in the r_{ik}, and so on, may be written in the form

$$\sum r_{ik} H_{lm} = 0, \qquad i, k, l, m \text{ all different}, \tag{5}$$

in which

$$H_{lm} \equiv \frac{\partial H}{\partial x_l}\frac{\partial H}{\partial y_m} - \frac{\partial H}{\partial x_m}\frac{\partial H}{\partial y_l}.$$

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† Snyder and Carroll-Rusk, *A series of involutorial transformations in S_n belonging multiply to a non-linear line complex,* American Journal of Mathematics, vol. 59 (1937), pp. 775–782.

The equation (5) is linear in x and in y and contains λ to degree m_1+m_2+2; it can be represented, for example, by $\sum A_i x_i=0$.

The coordinates f_i of the point F in which t meets g are given by the determinants of the third order in the matrix

$$\left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ A_1 & A_2 & A_3 & A_4 \end{matrix} \right\| .$$

Thus, for example,

$$f_1 = r_{23}A_4 + r_{34}A_2 + r_{42}A_3. \tag{6}$$

The coordinates f_i are linear in y and are of order $2(m_1+m_2+1)$ in λ. For any given λ the equations (6) define a harmonic homology.

The line t meets $H(y)$ again in the point y' defined by $\rho y_i' = \sigma f_i + \tau y_i$, in which $\sigma = H(y, f)$, $\tau = -H(f)$. The transformation $y \sim y'$ is a Cremona involutorial transformation I of S_3 into itself.

The surface $\sigma=0$ is the locus of invariant points of I. Since from the definition of I there is no such surface, σ must divide out of the transformation and hence be a factor of τ.

2. **Fundamental lines of the second kind.** For a given point F, the surface $\sigma=H(f, y)=0$ is the polar plane of F as to its associated quadric. When F is on H, σ is the tangent plane to H at F. Let g meet H at z and w. When F is at z, $H(z, y)=0$ is the tangent plane to H at z, which contains g'. If g' meets H at z' and w', zz' lies on H and meets g and g'; hence it is a fundamental line of the second kind, every point of it having the whole line for conjugate; similarly for zw', wz', ww'. The locus of these lines is the surface $\sigma=0$. Its equation is linear in y_i and in f_i, which contains y linearly and λ to multiplicity $2(m_1+m_2+1)$. Moreover, $H(f, y)$ involves λ explicitly to multiplicity 1; hence the surface is of order $4(m_1+m_2+2)$ and contains the base C_4 of $|H|$ to multiplicity $2(m_1+m_2)+3$.

Since τ is $H(f)$, it is quadratic in y and contains λ to multiplicity $4(m_1+m_2)+5$.

After removing the factor σ from τ, the quotient is of order $2(m_1+m_2+1)$ in λ and does not contain y. The reduced form of the equations of I is then $\rho y_i' = f_i + h y_i$, $h=\tau/\sigma$. The transformation is of order $4(m_1+m_2)+5$ and contains the base C_4 to multiplicity $2(m_1+m_2+1)$. Since each quadric of the pencil is transformed into itself by I, it follows that the principal surface M, conjugate to C_4, is of order $8(m_1+m_2+1)$ and contains C_4 to multiplicity $4(m_1+m_2)+3$.

3. **Isolated fundamental points.** For certain values of λ the line g touches its associated quadric H. They are given by the tact-invariant of H, obtained by bordering the discriminant of H by the two sets of elements a_i, b_i and equating to zero. It is of order $2(m_1+m_2+1)$ in λ. This tact-invariant is $h=\tau/\sigma$. The image of each point of contact P_i is the quadric H of the pencil which contains it. These points are therefore double on the web of surfaces, conjugates to the planes of space. They and C_4 are the only base elements of the web and completely define it.

A point on C_4 has for its conjugate a rational curve of order $2(m_1+m_2+1)$, expressed parametrically by $\rho x_i=f_i(\lambda)$, wherein y is replaced by a point ξ of C_4 and λ is the independent parameter. Then if ξ is eliminated by means of $H_1(\xi)=0$, $H_2(\xi)=0$, the equation $M=0$ results. The table of characteristics can now be written: If $m_1+m_2=m$, then

$$
\begin{aligned}
S_1 &\sim S_{4m+5}:\ C_4^{2(m+1)}2(m+1)P_i^2,\\
C_4 &\sim M_{8(m+1)}:\ C_4^{4m+3}2(m+1)P_i,\\
P_i &\sim H_{2,i}:\ C_4P_i,\\
J &\equiv M2(m+1)H_{2,i}^2.
\end{aligned}
$$

The locus of the points of intersection (g, H) lies on the ruled surface R, locus of g, on the locus $A=0$, generated by $(ax)=0$ and $H_1=\lambda H_2$, and also on $B=0$, generated by $(bx)=0$ and $H_1=\lambda H_2$. It is of order $2m+1$. Any point of the curve, which is always hyper-elliptic, is invariant under I. Similarly, the locus (g', H), which may be of different order, may be determined.

The congruence of bisecants of C_4 is transformed into itself.

4. **Case of m_1 or $m_2=0$.** If the line g lies in a plane, then one of its equations does not contain λ and the plane is invariant under I. The plane meets each g' in a point G', pole of g as to the associated conic; hence the plane involution I' can be defined by means of a pencil of conics and a projective rational curve. If the locus of G' is of order n, the table of characteristics is

$$
\begin{aligned}
C_1 &\sim C_{4n+3}:\ 4B^{2n+1}(2n+1)P_i^2,\\
B &\sim \beta_{2n+1}:\ B^{n+1}3B_k^n,\\
P_i &\sim C_2:\ 4BP_i,\\
\sigma_{2n+3} &\equiv \sigma:\ 4B^{n+1}(2n+1)P_i,\\
J &\equiv 4\beta_{2n+1}(2n+1)C_2.
\end{aligned}
$$

As is thus seen, this is a semi-symmetric involution of Ruffini.*

By considering the various types of pencils of conics and choosing points on the locus of G' in all possible relations as to the composite conics of the pencil, all the non-monoidal semi-symmetric series of Ruffini involutions can be obtained. Most of the theorems already appearing in the literature follow immediately from this point of view.

5. **Map of** I. For many problems it is desirable to have a map of an involution, such that a point x' of the map represents a point y and its conjugate under I. When x' describes a linear space S', I is said to be rational.

There are three systems of invariant surfaces. If the equations of I are $\rho y_i' \doteq \phi_i(y)$, these may be written:

$$\begin{aligned} \sum a_i y_i \phi_i &= 0, &\quad \infty^3 \text{ system}, \\ \sum a_i(y_i\phi_k - y_k\phi_i) &= 0, &\quad \infty^4 \text{ system}, \\ \sum a_i(y_i\phi_k + y_k\phi_i) &= 0, &\quad \infty^5 \text{ system}. \end{aligned}$$

Each system defines a map of the given involution. If it is assumed that $H_1(y) = x_1'$, $H_2(y) = x_2'$, by this scheme each quadric of the pencil is mapped upon a plane of a pencil; since by I each quadric of the pencil remains invariant, the pencil of planes furnishes an adequate map. If the space (y) is thought of as superposed on (x') with the same coordinate system and the same system of directrices g, g', the map has no surface of branch points since I contains no surface of invariant points.

The only fundamental points of the map are those of the axis of the pencil planes. The equations are

$$\rho x_i' = (H_2 y_1 - H_1 y_2) f_i - (H_2 f_1 - H_1 f_2) y_i.$$

The quadrics associated with contact on P_i are projected upon their image plane stereographically, the general quadric by a harmonic axial homology.

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* *Sulla risoluzione delle due equazioni di condizione delle trasformazioni Cremoniane delle figure piane*, Memorie dell'Accademia dell' Istituto, Bologna, (3), vol. 8 (1877), pp. 457–525. D. Montesano, *Le corrispondenze birazionali piane semisimmetriche*, Rendiconti dell'Accademia, Naples, (3), vol. 24 (1918), pp. 31–78.