ON FOURIER SERIES WITH RESTRICTED COEFFICIENTS*

OTTO SZÁSZ

1. **Introduction.** Consider a real-valued function $f(x)$, periodic with period 2π and Lebesgue integrable. Let

$$f(x) \sim \frac{a_0}{2} + \sum_1^\infty (a_\nu \cos \nu x + b_\nu \sin \nu x) \tag{1.1}$$

be its Fourier series, and let $s_0 = a_0/2$,

$$s_n(f;\, x) = s_n = \frac{a_0}{2} + \sum_1^n (a_\nu \cos \nu x + b_\nu \sin \nu x), \qquad n = 1, 2, 3, \cdots, \tag{1.2}$$

be its partial sums. We shall mainly restrict ourselves to series satisfying the conditions

$$na_n \geqq -p, \qquad nb_n \geqq -p, \qquad \text{for} \qquad n = 1, 2, 3, \cdots, \tag{1.3}$$

where $p \geqq 0$. We shall in particular consider the following problem: Suppose

$$-\mu \leqq f(x) \leqq \mu \quad \text{in} \quad -\pi < x < \pi; \tag{1.4}$$

then what is the best upper bound $C_n(\mu, p)$ for the partial sums $|s_n(f;\, x)| \leqq C_n(\mu, p)$, $(n \geqq 1)$, under the assumption (1.3)?

It is known that the sequence $\{C_n\}$ is bounded (cf. Szász [5], [6]);† hence l.u.b. $C_n(\mu, p) = C(\mu, p)$ is finite. For $p = 0$ the author [9] proved recently that

$$C(\mu, 0) < (2 + 4/\pi)\mu < 3.3\mu; \tag{1.5}$$

for $p > 0$ the sharpest estimates so far were given by Fekete [2] using a device of Paley and Fejér [1]. Fekete proved that

$$C(\mu, p) < 5\mu + 6p, \tag{1.6}$$

and also that

$$C(\mu, p) < 5\mu + 8(\mu p)^{1/2}. \tag{1.7}$$

---

* Presented to the Society, April 9, 1938.

† See the list of references at the end of this paper.

Note that for any $\rho > 0$, we have $s_n(\rho f; x) = \rho s_n(f; x)$, which gives the relation $\rho C_n(\mu, p) = C_n(\rho\mu, \rho p)$, $(p \geqq 0, \rho > 0)$, and in particular

$$C_n(\mu, p) = \mu C_n(1, p/\mu). \tag{1.8}$$

Hence in the discussion we may put $\mu = 1$; that is, $|f(x)| \leqq 1$. Using a similar tool as in [**9**] we shall improve upon the known results. We shall get sharper estimates for the kth partial sum assuming (1.3) only for $n = 1, 2, \cdots, 2k-1$.

2. **Certain identities and inequalities.** Given an infinite series $\sum_0^\infty u_\nu$ let $\sum_0^n u_\nu = U_n$, $\sum_0^{n-1} U_\nu = U_n^{(1)} = \sum_0^n (n-\nu)u_\nu$, $(n = 0, 1, 2, \cdots)$; we then have

$$\begin{aligned} \frac{1}{2n} U_{2n}^{(1)} - \frac{1}{n} U_n^{(1)} &= \frac{1}{2n}\left\{ \sum_0^{2n} (2n - \nu)u_\nu - 2\sum_0^n (n - \nu)u_\nu \right\} \\ &= \frac{1}{2n}\left\{ \sum_0^n \nu u_\nu + \sum_{n+1}^{2n} (2n - \nu)u_\nu \right\}, \end{aligned} \tag{2.1}$$

for $n = 1, 2, \cdots$, and $U_n^{(1)} = n\sum_0^n u_\nu - \sum_0^n \nu u_\nu = nU_n - \sum_0^n \nu u_\nu$, or

$$U_n = \frac{1}{n} U_n^{(1)} + \frac{1}{n}\sum_1^n \nu u_\nu, \qquad n = 1, 2, \cdots. \tag{2.2}$$

We get similarly

$$U_n = \frac{1}{2n} U_{2n}^{(1)} + \left(\frac{1}{2n} U_{2n}^{(1)} - \frac{1}{n} U_n^{(1)}\right) - \frac{1}{n}\sum_{n+1}^{2n} (2n - \nu)u_\nu, \tag{2.3}$$

for $n = 1, 2, \cdots$, and ([**4**], p. 186)

$$U_n = \frac{U_{n+\kappa}^{(1)}}{n+\kappa} + \frac{n}{\kappa}\left(\frac{U_{n+\kappa}^{(1)}}{n+\kappa} - \frac{U_n^{(1)}}{n}\right) - \frac{1}{\kappa}\sum_{\nu=n+1}^{n+\kappa} (n - \nu + \kappa)u_\nu, \qquad \kappa \geqq 1,\ n \geqq 1. \tag{2.4}$$

This reduces to (2.3) for $\kappa = n$, whereas for $\kappa = 1$ it yields

$$U_n = \frac{1}{n+1} U_{n+1}^{(1)} + \frac{1}{n+1}\sum_1^n \nu u_\nu. \tag{2.5}$$

We are using in addition the fact that the assumption

$$-1 \leqq f(x) \leqq 1 \quad \text{for} \quad -\pi < x < \pi \tag{2.6}$$

involves

$$\left|\sum_0^{n-1} s_\nu(x)\right| \leqq n, \qquad n = 1, 2, \cdots, \tag{2.7}$$

which is the classical result of Fejér, and (compare [9]) the relations

$$
(2.8)\qquad \begin{aligned} &\left| \sigma_{2n}(x) - \sigma_n(x) \right| \leqq 2/\pi, \\ &\left| \bar\sigma_{2n}(x) - \bar\sigma_n(x) \right| \leqq 2/\pi, \qquad n = 1, 2, \cdots, \end{aligned}
$$

where $\sigma_n = (1/n)\sum_0^{n-1} s_\nu$, and $\bar\sigma_n$ is the corresponding mean for the conjugate series. In view of (2.1) the last inequalities can be written as

$$
(2.9)\qquad \begin{aligned} &\Bigl| \sum_1^n \nu(a_\nu \cos \nu x + b_\nu \sin \nu x) \\ &\qquad + \sum_{n+1}^{2n} (2n - \nu)(a_\nu \cos \nu x + b_\nu \sin \nu x) \Bigr| \leqq \frac{4}{\pi} n, \\ &\Bigl| \sum_1^n \nu(b_\nu \cos \nu x - a_\nu \sin \nu x) \\ &\qquad + \sum_{n+1}^{2n} (2n - \nu)(b_\nu \cos \nu x - a_\nu \sin \nu x) \Bigr| \leqq \frac{4}{\pi} n. \end{aligned}
$$

We shall use also the following more general inequalities:

$$
\begin{aligned} &\Bigl| \sum_{\nu=1}^n \nu\left\{ a_{\nu+\lambda-1} \cos (\nu+\lambda-1)x + b_{\nu+\lambda-1} \sin (\nu+\lambda-1)x \right\} \\ &+ \sum_{\nu=n+1}^{2n} (2n-\nu)\left\{ a_{\nu+\lambda-1} \cos (\nu+\lambda-1)x + b_{\nu+\lambda-1} \sin (\nu+\lambda-1)x \right\} \Bigr| \leqq \frac{4}{\pi} n, \\ &\Bigl| \sum_{\nu=1}^n \nu\left\{ b_{\nu+\lambda-1} \cos (\nu+\lambda-1)x - a_{\nu+\lambda-1} \sin (\nu+\lambda-1)x \right\} \\ &+ \sum_{\nu=n+1}^{2n} (2n-\nu)\left\{ b_{\nu+\lambda-1} \cos (\nu+\lambda-1)x - a_{\nu+\lambda-1} \sin (\nu+\lambda-1)x \right\} \Bigr| \leqq \frac{4}{\pi} n, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \lambda \geqq 1 - n/2. \end{aligned}
$$

3. **Two auxiliary theorems.** We prove the following lemma:

LEMMA 1. *The sequence* $\rho_n = n^{-1}\sum_{n+1}^{2n}(2n-\nu)/\nu$, $(n=1, 2, 3, \cdots)$, *is monotonically increasing, and* $\rho_n \uparrow 2 \log 2 - 1 = 0.38629 \cdots$.

For the proof note that

$$
\begin{aligned} \rho_{n+1} - \rho_n &= 2\left(\sum_{n+2}^{2n+2} \frac{1}{\nu} - \sum_{n+1}^{2n} \frac{1}{\nu} \right) = 2\left(\frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \right) \\ &= \frac{2}{2n+1} - \frac{1}{n+1} > 0; \end{aligned}
$$

furthermore

$$\rho_n = \frac{1}{n}\sum_{n+1}^{2n}\frac{2-\nu/n}{\nu/n} \to \int_1^2 \frac{2-x}{x}\,dx = 2\log 2 - 1.$$

LEMMA 2. *The sequence* $\tau_n = n^{-1}\sum_n^{2n-1}(\nu-n)/\nu$, $(n=1, 2, 3, \cdots)$, *is monotonically increasing, and*

$$\tau_n \uparrow 1 - \log 2 = 0.30685\cdots.$$

Indeed

$$\tau_{n+1} - \tau_n = \sum_n^{2n-1}\frac{1}{\nu} - \sum_{n+1}^{2n+1}\frac{1}{\nu} = \frac{1}{n} - \frac{1}{2n} - \frac{1}{2n+1} = \frac{1}{2n} - \frac{1}{2n+1} > 0$$

and

$$\tau_n = \frac{1}{n}\sum_n^{2n-1}\frac{\nu/n - 1}{\nu/n} \to \int_1^2 \frac{x-1}{x}\,dx = 1 - \log 2.$$

4. **Estimates for $\sum \nu|a_\nu|$, $\sum \nu|b_\nu|$, and related sums.** In view of (2.1) the inequalities (2.8) can be written as

$$\left|\sum_1^n \nu u_\nu + \sum_{n+1}^{2n}(2n-\nu)u_\nu\right| \leqq \frac{4}{\pi}n, \qquad n = 1, 2, \cdots,$$

where u_ν is $a_\nu \cos \nu x + b_\nu \sin \nu x$ or $b_\nu \cos \nu x - a_\nu \sin \nu x$, respectively. In particular, for $x=0$, we have $\sum_1^n \nu a_\nu + \sum_{n+1}^{2n+1}(2n-\nu)a_\nu \leqq 4n/\pi$, $(n \geqq 2)$, and $|a_1| \leqq 4/\pi$; hence

$$\sum_1^n(\nu a_\nu + p) + \sum_{n+1}^{2n-1}(2n-\nu)\left(a_\nu + \frac{p}{\nu}\right) \leqq n\left(\frac{4}{\pi} + p\right) + p\sum_{n+1}^{2n-1}\frac{2n-\nu}{\nu}, \qquad n \geqq 2. \tag{4.1}$$

We now assume $p > 0$ and

$$\nu a_\nu \geqq -p, \qquad \nu = 1, 2, \cdots, 2n-1; \tag{4.2}$$

then (4.1) gives

$$\sum_1^n(\nu a_\nu + p) \leqq n\left\{\frac{4}{\pi} + p + \frac{p}{n}\sum_{n+1}^{2n-1}\frac{2n-\nu}{\nu}\right\},$$

and, using Lemma 1, we obtain $\sum_1^n \nu a_\nu < n\{4/\pi + p(2\log 2 - 1)\}$, $(n \geqq 2)$. Also from (4.2) $\nu(|a_\nu| - a_\nu) \leqq 2p$, $(\nu = 1, 2, \cdots, 2n-1)$; hence $\sum_1^n \nu|a_\nu| < n\{4/\pi + p(1+2\log 2)\}$, $(n \geqq 1)$. In the same way the assumption $\nu b_\nu \geqq -p$, $(\nu = 1, 2, \cdots, 2n-1)$, yields $\sum_1^n \nu|b_\nu| < n\{4/\pi + p(1+2\log 2)\}$. A fortiori

$$\left|\sum_1^n \nu a_\nu \cos \nu x\right| < n\left\{\frac{4}{\pi} + p(1 + 2\log 2)\right\}, \qquad n \geqq 1, \tag{4.3}$$

and

$$\left|\sum_1^n \nu b_\nu \sin \nu x\right| < n\left\{\frac{4}{\pi} + p(1 + 2\log 2)\right\}, \qquad n \geqq 1. \tag{4.4}$$

Again from (4.1)

$$\sum_{n+1}^{2n-1} (2n - \nu)\left(a_\nu + \frac{p}{\nu}\right) \leqq n\left(\frac{4}{\pi} + p\right) + p\sum_{n+1}^{2n-1} \frac{2n - \nu}{\nu}, \qquad n \geqq 2;$$

hence $\sum_{n+1}^{2n-1}(2n-\nu)a_\nu \leqq n(4/\pi+p)$. But from (4.2) we have $(2n-\nu)\cdot(|a_\nu|-a_\nu) \leqq 2p(2n-\nu)/\nu$; hence

$$\sum_{n+1}^{2n-1} (2n - \nu) \mid a_\nu \mid \leqq n\left(\frac{4}{\pi} + p\right) + 2p\sum_{n+1}^{2n-1} \frac{2n - \nu}{\nu} \cdot \tag{4.5}$$

Using again Lemma 1, we get

$$\sum_{n+1}^{2n-1} (2n - \nu) \mid a_\nu \mid < n\left\{\frac{4}{\pi} + p(4\log 2 - 1)\right\}, \tag{4.6}$$

and similarly $\sum_{n+1}^{2n-1}(2n-\nu)|b_\nu| < n\{4/\pi+p(4\log 2-1)\}$. Summarizing, we have the following theorem:

THEOREM 1. *Suppose* $|f(x)| \leqq 1$; *if in addition*

$$\nu a_\nu \geqq -p, \qquad \nu = 1, 2, \cdots, 2n-1;\ p > 0;\ n > 0, \tag{4.7}$$

then, for $\kappa = 1, \cdots, n$,

$$\sum_1^\kappa \nu \mid a_\nu \mid < \kappa\left\{\frac{4}{\pi} + p(1 + 2\log 2)\right\}, \tag{4.8}$$

$$\sum_{\nu=\kappa+1}^{2\kappa} (2\kappa - \nu) \mid a_\nu \mid < \kappa\left\{\frac{4}{\pi} + p(4\log 2 - 1)\right\}. \tag{4.9}$$

Similarly, if

$$\nu b_\nu \geqq -p, \qquad \nu = 1, \cdots, 2n-1;\ p > 0;\ n > 0, \tag{4.10}$$

then

$$\sum_1^\kappa \nu \mid b_\nu \mid < \kappa\left\{\frac{4}{\pi} + p(1 + 2\log 2)\right\}, \tag{4.11}$$

$$\sum_{\kappa+1}^{2\kappa} (2\kappa - \nu) \mid b_\nu \mid < \kappa\left\{\frac{4}{\pi} + p(4\log 2 - 1)\right\}. \tag{4.12}$$

For $\kappa = n = 1$ we have also the inequalities $|a_1| \leqq 4/\pi$, $|b_1| \leqq 4/\pi$. Note that $4/\pi = 1.2732 \cdots$, $1+2\log 2 = 2.3862 \cdots$, $4\log 2 - 1 = 1.7725 \cdots$.

5. **Estimates for partial sums.** We now assume (4.7) and (4.10); applying Theorem 1, formula (2.2), and the inequality (2.7) we obtain

$$\left| \frac{a_0}{2} + \sum_1^\kappa a_\nu \cos \nu x \right| < 1 + \frac{4}{\pi} + p(1 + 2 \log 2), \tag{5.1}$$

for $\kappa = 1, \cdots, n$, and

$$\left| \sum_1^\kappa b_\nu \sin \nu x \right| < 1 + \frac{4}{\pi} + p(1 + 2 \log 2), \qquad \kappa = 1, \cdots, n. \tag{5.2}$$

Using again (2.2) with $u_\nu = a_\nu \cos \nu x + b_\nu \sin \nu x$ we obtain $|s_\kappa(f; x)| < 1 + 8/\pi + 2p(1+2 \log 2)$, $(\kappa = 1, \cdots, n)$. Using now (1.8) and the assumption (1.4) we get

$$|s_\kappa(f; x)| < (1 + 8/\pi)\mu + 2p(1 + 2 \log 2) < 3.547\mu + 4.773p.$$

This is sharper than (1.6).

Note that the formula $s_0 = a_0/2 = (2\pi)^{-1}\int_{-\pi}^{\pi} f(x)dx$ gives $|s_0| \leqq \mu$, assuming (1.4) only. In a similar way formulas (2.3), (2.8), (4.9), and (4.12) give

$$\left| \frac{a_0}{2} + \sum_1^\kappa a_\nu \cos \nu x \right| < 1 + \frac{6}{\pi} + p(4 \log 2 - 1), \tag{5.3}$$

$$\left| \sum_1^\kappa b_\nu \sin \nu x \right| < 1 + \frac{6}{\pi} + p(4 \log 2 - 1), \tag{5.4}$$

$$|s_\kappa(f; x)| < 1 + \frac{10}{\pi} + 2p(4 \log 2 - 1),$$

$$\kappa = 1, \cdots, n.$$

Since $1 + 10/\pi = 4.18309 \cdots$, $8 \log 2 - 2 = 3.54518 \cdots$, this estimate again is sharper than (1.6).

Using again formula (1.8) and summarizing, we have the following theorem:

Theorem 2. *If* $|f(x)| \leqq \mu$ *and*

$$\nu a_\nu \geqq -p, \qquad \nu b_\nu \geqq -p, \qquad \nu = 1, \cdots, 2n-1; p > 0, \tag{5.5}$$

then for $\kappa = 1, \cdots, n$

$$\left| \frac{a_0}{2} + \sum_1^\kappa (a_\nu \cos \nu x + b_\nu \sin \nu x) \right| < \begin{cases} (1+\ 8/\pi)\mu + 2p(1+2 \log 2) \\ (1+10/\pi)\mu + 2p(4 \log 2 - 1). \end{cases}$$

6. **Estimates for $\sum_{\nu=l+1}^{2l-1}(2l-\nu)|a_{\nu+\lambda-1}|$ and related sums.** In [9] the author proved, under the assumption (2.6), that

$$\left|\sum_{\nu=1}^{l} \nu u_{\nu+\lambda-1} + \sum_{\nu=l+1}^{2l} (2l-\nu)u_{\nu+\lambda-1}\right| \leqq \frac{4}{\pi} l, \qquad \lambda = 1, 2, 3, \cdots; l \geqq 1;$$

here u_ν stands for $a_\nu \cos \nu x + b_\nu \sin \nu x$ or $b_\nu \cos \nu x - a_\nu \sin \nu x$. Hence, for $l > 1, \sum_{\nu=1}^{l} \nu a_{\nu+\lambda-1} + \sum_{\nu=l+1}^{2l-1}(2l-\nu)a_{\nu+\lambda-1} \leqq 4l/\pi$, and for $l=1$, $|u_\lambda| \leqq 4/\pi$, $(\lambda = 1, 2, 3, \cdots)$. Thus for $l>1$

$$\begin{aligned}(6.1)\qquad &\sum_{\nu=1}^{l} \nu\left(a_{\nu+\lambda-1} + \frac{p}{\nu+\lambda-1}\right) + \sum_{\nu=l+1}^{2l-1}(2l-\nu)\left(a_{\nu+\lambda-1} + \frac{p}{\nu+\lambda-1}\right) \\ &\qquad \leqq \frac{4}{\pi} l + p\left\{\sum_{\nu=1}^{l} \frac{\nu}{\nu+\lambda-1} + \sum_{\nu=l+1}^{2l-1} \frac{2l-\nu}{\nu+\lambda-1}\right\}.\end{aligned}$$

We now assume for a given integer N that

$$(6.2) \qquad \nu a_\nu \geqq -p, \qquad \nu = 1, 2, \cdots, N,$$

and suppose that

$$(6.3) \qquad 2l + \lambda - 2 \leqq N;$$

then from (6.1), for $l>1$, we have

$$(6.4) \qquad \sum_{\nu=l+1}^{2l-1} (2l-\nu)a_{\nu+\lambda-1} \leqq \frac{4}{\pi} l + p\sum_{\nu=1}^{l} \frac{\nu}{\nu+\lambda-1}.$$

Furthermore $|a_\nu| - a_\nu \leqq 2p/\nu$, $(\nu = 1, \cdots, N)$; hence using (6.4) we obtain

$$\sum_{\nu=l+1}^{2l-1} (2l-\nu)\,|a_{\nu+\lambda-1}| \leqq \frac{4}{\pi} l + p\left\{2\sum_{\nu=l+1}^{2l-1} \frac{2l-\nu}{\nu+\lambda-1} + \sum_{\nu=1}^{l} \frac{\nu}{\nu+\lambda-1}\right\}.$$

Similarly, assuming

$$(6.5) \qquad \nu b_\nu \geqq -p, \nu = 1, \cdots, N,$$

we get, for $l>1$,

$$\sum_{\nu=l+1}^{2l-1} (2l-\nu)\,|b_{\nu+\lambda-1}| \leqq \frac{4}{\pi} l + p\left\{2\sum_{\nu=l+1}^{2l-1} \frac{2l-\nu}{\nu+\lambda-1} + \sum_{\nu=1}^{l} \frac{\nu}{\nu+\lambda-1}\right\}.$$

A simple calculation yields

$$\sum_{\nu=l+1}^{2l-1} \frac{2l-\nu}{\nu+\lambda-1} \leqq \frac{1}{l+\lambda} \sum_{\nu=l+1}^{2l-1} (2l-\nu) = \frac{l(l-1)}{2(l+\lambda)},$$

$$\sum_{\nu=1}^{l} \frac{\nu}{\nu+\lambda-1} < \frac{l^2}{l+\lambda-1}.$$

We have thus proved the following theorem:

THEOREM 3. *Under the assumptions* (2.6), (6.2), (6.3), *and* (6.5) *we have, for* $l>1$, $\lambda=1, 2, 3, \cdots$,

$$(6.6) \qquad \sum_{\nu=l+1}^{2l} (2l-\nu)v_{\nu+\lambda-1} < \left\{\frac{4}{\pi} + p\left(\frac{l-1}{l+\lambda} + \frac{l}{l+\lambda-1}\right)\right\} l,$$

where v_ν *stands for* $|a_\nu|$ *or* $|b_\nu|$.

Formula (6.6) remains true also for $l=1$.

7. **Further estimates for partial sums.** Assuming (2.6) we get, from (2.4) and (2.7),

$$(7.1) \qquad |U_n| \leqq 1 + \frac{2n}{\kappa} + \frac{1}{\kappa}\left|\sum_{\nu=n+1}^{n+\kappa}(n-\nu+\kappa)u_\nu\right|, \qquad \kappa \geqq 1,\ n \geqq 1,$$

where u_ν means $a_\nu \cos \nu x$, $b_\nu \sin \nu x$, or $a_\nu \cos \nu x + b_\nu \sin \nu x$. Replacing in (6.6) l by κ and λ by $n-\kappa+1$, we get

$$\sum_{\nu=l+1}^{2l}(2l-\nu)v_{\nu+\lambda-1} = \sum_{\nu=\kappa+1}^{2\kappa}(2\kappa-\nu)v_{\nu+n-\kappa} \equiv \sum_{\nu=n+1}^{n+\kappa}(n-\nu+\kappa)v_\nu;$$

hence, using (6.6) and (7.1), we obtain

$$(7.2) \qquad |U_n| < 1 + \frac{2n}{\kappa} + \frac{4}{\pi} + p\left(\frac{\kappa-1}{n+1} + \frac{\kappa}{n}\right), \qquad n \geqq \kappa \geqq 1.$$

The condition (6.3) becomes

$$(7.3) \qquad n + \kappa - 1 \leqq N.$$

Here U_n means $a_0/2 + \sum_1^n a_\nu \cos \nu x$ or $\sum_1^n b_\nu \sin \nu x$. We observe that $2n/\kappa + p\{(\kappa-1)/(n+1) + \kappa/n\}$ decreases with increasing κ where $\kappa(\kappa+1) \leqq 2n^2(n+1)/p(2n+1)$. We thus put

$$(7.4) \qquad \kappa = \left[\frac{1}{2} + \left\{\frac{1}{4} + \frac{2n^2(n+1)}{p(2n+1)}\right\}^{1/2}\right];$$

obviously

$$\kappa > -\frac{1}{2} + \left\{\frac{1}{4} + \frac{2n^2(n+1)}{p(2n+1)}\right\}^{1/2} \geqq 0;$$

hence $\kappa \geqq 1$. We now get easily the relation

$$\frac{2n}{\kappa}+p\frac{\kappa}{n}<\frac{2n}{-\frac{1}{2}+\left\{\frac{1}{4}+\frac{2n^2(n+1)}{p(2n+1)}\right\}^{1/2}}+\frac{p}{n}\left(-\frac{1}{2}+\left\{\frac{1}{4}+\frac{2n^2(n+1)}{p(2n+1)}\right\}^{1/2}\right),$$

and

$$\frac{2n}{\kappa}+p\frac{\kappa}{n}+p\frac{\kappa-1}{n+1}<\frac{2p(2n+1)}{n(n+1)}\left\{\frac{1}{4}+\frac{2n^2(n+1)}{p(2n+1)}\right\}^{1/2}\leqq 4p^{1/2}$$

if

$$(7.5)\qquad p\leqq 16n/9,\qquad p\leqq 2n^4/(2n+1)^2.$$

On putting $N=2n-1$ in (7.3) we get $\kappa\leqq n$; in view of (7.4) this condition is satisfied if

$$(7.6)\qquad p\geqq\frac{2n(n+1)}{(2n+1)(n-1)},\qquad n>1.$$

Thus with the restrictions (7.5) and (7.6) the assumptions (2.6) and (5.5) imply

$$(7.7)\qquad |U_n|<1+4/\pi+4p^{1/2}.$$

In the cases $p>16n/9$ or $p>2n^4/(2n+1)^2$, (7.7) can be deduced from $a_0^2/2+\sum_1^n(a_\nu^2+b_\nu^2)\leqq\pi^{-1}\int_{-\pi}^{\pi}f^2(x)dx\leqq 2$. Now

$$\frac{1}{2}|a_0|+\sum_1^n|a_\nu|\leqq\left\{\frac{1}{2}a_0^2+\sum_1^n a_\nu^2\right\}^{1/2}\left(\frac{1}{2}+n\right)^{1/2}$$
$$\leqq(1+2n)^{1/2}<1+4/\pi+4p^{1/2}.$$

Finally in the case $p<2n(n+1)/(2n+1)(n-1)$ we can deduce (7.7) from (2.5), (2.7), (4.8), and (4.11), which give $|U_n|<1+[n/(n+1)]\cdot\{4/\pi+p(1+2\log 2)\}$ and this is in the present case less than $1+4/\pi+4p^{1/2}$ for $n>1$, since $[n/(n+1)]p^{1/2}<4/(1+2\log 2)$. Summarizing and using (1.8), we complete the proof of the following theorem:

THEOREM 4. *Under the assumptions* (1.4) *and* (5.5),

$$\left|\frac{a_0}{2}+\sum_1^\kappa a_\nu\cos\nu x\right|<\left(1+\frac{4}{\pi}\right)\mu+4(p\mu)^{1/2},$$
$$\left|\sum_1^\kappa b_\nu\sin\nu x\right|<\left(1+\frac{4}{\pi}\right)\mu+4(p\mu)^{1/2},\quad\kappa=1,2,\cdots,n.$$

In the case $u_\nu = a_\nu \cos \nu x + b_\nu \sin \nu x$, we get from (7.1) and (6.6) in a similar way

$$|U_n| < 1 + \frac{2n}{\kappa} + \frac{8}{\pi} + 2p\left(\frac{\kappa - 1}{n + 1} + \frac{\kappa}{n}\right), \qquad 1 \leqq \kappa \leqq n.$$

We now put

$$\kappa = \left[\frac{1}{2} + \left\{\frac{1}{4} + \frac{n^2(n+1)}{p(2n+1)}\right\}^{1/2}\right];$$

that is, in our previous argument p is replaced by $2p$. We get the following theorem:

THEOREM 5. *Under the assumptions* (1.4) and (5.5),

$$|s_\kappa(f; x)| < \left(1 + \frac{8}{\pi}\right)\mu + 4(2p\mu)^{1/2}, \qquad \kappa = 1, \cdots, n.$$

Since $1+8/\pi = 3.5464 \cdots < 5$, and $4(2)^{1/2} < 8$, our estimate is sharper than (1.7).

The results can be extended to almost periodic functions (for $p=0$ see [**3**], [**7**], [**8**], [**9**]) and to Fourier integrals; also the condition (1.3) can be generalized to "slow oscillation," improving upon some results of Fekete and the author.

REFERENCES

1. L. Fejér, *On a theorem of Paley*, this Bulletin, vol. 40 (1934), pp. 469–475.

2. M. Fekete, *Proof of three propositions of Paley*, ibid., vol. 41 (1935), pp. 138–144.

3. ———, *Some generalizations of Paley's theorem on Fourier series with positive coefficients*, Transactions of this Society, vol. 38 (1935), pp. 237–249.

4. O. Szász, *Zur Konvergenztheorie der Fourierschen Reihen*, Acta Mathematica, vol. 61 (1933), pp. 185–201.

5. ———, *A Fourier-féle sor részletösszegeinck korlátosságáról és összetartásáról*, Matematikai és Természettudományi Értesítö, vol. 50 (1933), pp. 125–146.

6. ———, *Convergence properties of Fourier series*, Transactions of this Society, vol. 37 (1935), pp. 483–500.

7. ———, *On the partial sums of certain Fourier series*, American Journal of Mathematics, vol. 59 (1937), pp. 696–708.

8. ———, *Fourier-sorok részletösszegeiröl*, Matematikai és Természettudományi Értesítö, vol. 56 (1937), pp. 382–396.

9. ———, *Some extremum problems in the theory of Fourier series*, to appear in the American Journal of Mathematics.

UNIVERSITY OF CINCINNATI