ON THE ABSOLUTE SUMMABILITY OF FOURIER SERIES*

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1. **Introduction.** A series $\sum u_n$ is said to be absolutely summable by a method $\mathfrak{a}$ defined by a matrix a_{mn} if

$$\sum_{m=1}^{\infty} \left| S_m(\mathfrak{a}, u) - S_{m-1}(\mathfrak{a}, u) \right| < \infty,$$

where

$$S_m(\mathfrak{a}, u) = \sum_{n=0}^{\infty} a_{mn} u_n.$$

Similarly a series is said to be absolutely summable $|A|$ if

$$S(r, u) = \sum_{n=0}^{\infty} u_n r^n \subset BV \quad \text{on} \quad (0, 1).$$

It is known that if $\sum u_n$ is absolutely summable $|C_\alpha|$ for some $\alpha > 0$, then it is absolutely summable $|A|$. There are, however, series absolutely summable $|A|$ but not $|C_\alpha|$ for any α whatever. We intend to give here an example of a Fourier series with that property.

Bosanquet† has proved that, if the Fourier series of $f(x)$ is absolutely summable $|C_\alpha|$, then the function

$$\phi_\beta(f, t) = \beta t^{-\beta} \int_0^t \left\{ f(x+\tau) + f(x-\tau) - 2f(x) \right\} (t-\tau)^{\beta-1} d\tau$$

is of bounded variation on $(0, \pi)$ for $\beta > \alpha$; and conversely, if $\phi_\alpha(t)$ is of bounded variation, the Fourier series of $f(x)$ is absolutely summable $|C_\beta|$, $(\beta > \alpha + 1)$.

2. **Preliminary definitions.** Let α_{nk}, β_{nk} be defined for $n = 1, 2, \cdots$, $k = 1, 2, \cdots, n$, by

$$(1) \qquad \alpha_{nk} = 2^{-k-n-n/(k-1/2)}, \qquad \beta_{nk} = 2^{-n} - 2^{-n-n/(k-1/2)}.$$

Then, since $k \leqq n$, we have

$$\beta_{nk} > 2^{-n-1}.$$

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† L. S. Bosanquet, *The absolute Cesàro summability of Fourier series*, Proceedings of the London Mathematical Society, vol. 41 (1936), pp. 517–528.

Let $f_{nk}(x)$ be defined over $-\pi \leqq x \leqq \pi$, so that

(2) $f_{nk}(x) = 2^n, \quad \beta_{nk} \leqq |x| \leqq \beta_{nk} + \alpha_{nk},$

(3) $f_{nk}(x) = -f_{nk}(x - 2^j\alpha_{nk}), \quad \beta_{nk} + 2^j\alpha_{nk} < |x| \leqq \beta_{nk} + 2^{j+1}\alpha_{nk},$

(4) $f_{nk}(x) = 0 \quad$ elsewhere on $\quad (-\pi, \pi),$

where in (3) j takes on the values $0, \cdots, k-1$. The relation (3) implies that

$$\int_{\beta_{nk}}^{\beta_{nk}+2\alpha_{nk}} f_{nk}(x)dx = \int_{\beta_{nk}}^{\beta_{nk}+\alpha_{nk}} f_{nk}(x)dx - \int_{\beta_{nk}}^{\beta_{nk}+\alpha_{nk}} f_{nk}(x)dx = 0,$$

and by induction

$$\int_{\beta_{nk}}^{\beta_{nk}+2^j\alpha_{nk}} f_{nk}(x)dx = 0, \qquad 1 \leqq j \leqq k.$$

If we define

$$\Phi_1(f, t) = \int_0^t \{f(x) + f(-x) - 2f(0)\} dx,$$

we can obtain the following relations analogous to (3) and (4):

(5) $\Phi_1(f_{nk}, t) = -\Phi_1(f_{nk}, t - 2^{j+1}\alpha_{nk}), \quad \beta_{nk} + 2^{j+1}\alpha_{nk} < x \leqq \beta_{nk} + 2^{j+2}\alpha_{nk},$

(6) $\Phi_1(f_{nk}, t) = 0 \quad$ elsewhere on $\quad (0, \pi),$

where in (5) j takes on the values $0, \cdots, k-2$.

We define by induction the functions

$$\Phi_{r+1}(f, t) = (r+1)\int_0^t \Phi_r(f, x)dx,$$

for which it can be shown by similar reasoning that, for $r \leqq k$,

(7) $\Phi_r(f_{nk}, t) = -\Phi_r(f_{nk}, t - 2^{r+j}\alpha_{nk}), \quad \beta_{nk} + 2^{r+j}\alpha_{nk} < x \leqq \beta_{nk} + 2^{r+j+1}\alpha_{nk},$

(8) $\Phi_r(f_{nk}, t) = 0, \quad$ elsewhere on $\quad (0, \pi),$

where in (7) j takes on the values $0, \cdots, k-r-1$. We notice that, at $x=0$, $\phi_r(f, t) = t^{-r}\Phi_r(f, t)$, and therefore for $r \leqq k-1$

$$\begin{aligned} \phi_r(f_{nk}, \beta_{nk} + \alpha_{nk}) &= 2r(\beta_{nk} + \alpha_{nk})^{-r} \int_{\beta_{nk}}^{\beta_{nk}+\alpha_{nk}} 2^n(\beta_{nk} + \alpha_{nk} - x)^{r-1}dx \\ &= 2^{n+1}r(\beta_{nk} + \alpha_{nk})^{-r} \int_0^{\alpha_{nk}} (\alpha_{nk} - x)^{r-1}dx \\ &> 2^{n+1}r2^{nr}\alpha_{nk}^r > 2^{-kr}2^{n/2k}, \end{aligned}$$

since

$$n(r+1) - r\{n + n/(k-1/2)\} = n\{1 - r/(k-1/2)\} > n/2k.$$

This shows that, for $r<k$,

$$T.V._{(0,\pi)}\phi_r(f_{nk}, x) > 2^{-kr}2^{n/2k}.$$

On the other hand,

$$\phi_k'(f_{nk}, t) = kt^{-k}\Phi_{k-1}(f_{nk}, t) - kt^{-k-1}\Phi_k(f_{nk}, t),$$

so that, if $I=(\beta_{nk}, \beta_{nk}+2^k\alpha_{nk})=(\beta_{nk}, 2^{-n})$, then

$$\begin{aligned}\int_I \left| \phi_k'(f_{nk}, t) \right| dt &= O\left\{2^{kn}\int_0^{2^k\alpha_{nk}} 2^n(2^k\alpha_{nk} - t)^{k-1}dt\right.\\ &\qquad \left. + 2^{(k+1)n}\int_0^{2^k\alpha_{nk}} 2^n(2^k\alpha_{nk} - t)^k dt\right\}\\ &= O(2^{-n/2k}).\end{aligned}$$

Therefore

$$T.V._{(0,\pi)}\phi_k(f_{nk}, t) = O(2^{-n/2k}).$$

3. **Definition of** $f(x)$. We define the functions

$$f_k(x) = \sum_{[\log_2 k]+1}^{\infty} f_{2^n+k,k}(x).$$

For $r \leqq k$

$$\phi_r(f_{2^n+k,k}, t)\cdot\phi_r(f_{2^m+k,k}, t) = 0, \qquad m \neq n,$$

and therefore

$$T.V._{(0,\pi)}\phi_r(f_k, t) = \sum_{[\log_2 k]+1}^{\infty} T.V._{(0,\pi)}\phi_r(f_{2^n+k,k}, t) = \infty, \quad r<k,$$

and

$$\begin{aligned}(9)\qquad T.V._{(0,\pi)}\phi_k(f_k, t) &= \sum_{[\log_2 k]+1}^{\infty} T.V._{(0,\pi)}\phi_k(f_{2^n+k,k}, t)\\ &= O\left(\sum_1^{\infty} 2^{-j/2k}\right) = O(1).\end{aligned}$$

It follows then that, for $s>k$,

$$\phi_s(f_k, t) \subset BV \quad \text{on} \quad (0, \pi).$$

The Fourier series of $f_k(x)$ must be absolutely summable $|A|$, at $x=0$. We set

$$A_k = T.V._{(0,1)}\frac{1}{\pi}\int_0^\pi f_k(t)\frac{1-r^2}{1-2r\cos t+r^2}dt.$$

A sequence d_k is then defined so that

$$d_k \leqq A_k 2^{-k}, \tag{10}$$

$$d_k \leqq 2^{-k}\int_0^\pi |f_k(x)|\,dx. \tag{11}$$

The function

$$f(x) = \sum_1^\infty d_k f_k(x)$$

is the one we set out to construct. By (11), $f(x) \subset L$, since

$$\int_0^\pi |f(x)|\,dx \leqq \sum_1^\infty d_k \int_0^\pi |f_k(x)|\,dx \leqq \sum_1^\infty 2^{-k} = 1.$$

We have, by (10),

$$T.V._{(0,1)}\frac{1}{\pi}\int_0^\pi f(t)\frac{1-r^2}{1-2r\cos t+r^2}dt \leqq \sum_1^\infty d_k A_k \leqq \sum_1^\infty 2^{-k} = 1,$$

which means that the Fourier series of $f(x)$ is absolutely summable $|A|$, at $x=0$. Finally, using (9) we see that

$$T.V._{(0,\pi)}\phi_j(f,t) > T.V._{(0,\pi)}\phi_j(f_j,t) - \left|\sum_1^{j-1} T.V._{(0,\pi)}\phi_j(f_k,t)\right|$$
$$= \infty - O(1) = \infty;$$

so the Fourier series of $f(x)$ cannot be $|C_j|$ summable at $x=0$, for any j. This completes the proof of our assertion.

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