FORMAL SYNTHESIS OF TWO PERIODIC CORRESPONDENCES, OF PERIOD FIVE AND SEVEN, RESPECTIVELY*

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Since the illumination of binary doubly quadratic forms by the theory of elliptic functions, it has been generally the custom to study them in no other light. Even Poncelet's picture of two circles, or a pencil of circles, receives scant attention from the general student. This situation resembles a monopoly and is adverse to the progress of diversified industries. A (2, 2) correspondence between variables (x) and (y) creates sequences of values, each of which is considered first as a y, then as an x. I shall consider one such relation, one which is symmetric and periodic. Its period shall be five.

Denote by $\Phi=0$, or $\Phi(x, y)=\Phi(y, x)=0$, a (2, 2) correspondence, symmetric in x and y. For convenience, we may regard it sometimes as homogeneous in x_1 and x_2, sometimes in y_1 and y_2, or at will as non-homogeneous.

Being symmetric, it has a unique representation quadratic in the two combinations $(x+y)$ and xy, or homogeneous in the three, x_1y_1, $(x_1y_2+x_2y_1)$, x_2y_2. If these combinations be taken as line-coordinates while x, y are point parameters on a conic, the corresponding equation $\Phi=0$ becomes the line-equation of a second conic, and we have Poncelet's picture.

Let five quantities, p, q, r, s, t, constitute a closed series or cycle, each one with its successor forming a pair x, y which satisfies the equation $\Phi=0$. As that symmetric equation has only five constants, it is completely determined by these five pairs. Hence we obtain

$$(1)\qquad F(x, y) = \begin{vmatrix} p^2q^2 & pq(p+q) & (p+q)^2 & pq & p+q & 1 \\ q^2r^2 & qr(q+r) & \cdot & \cdot & \cdot & 1 \\ r^2s^2 & \cdot & \cdot & \cdot & \cdot & 1 \\ s^2t^2 & \cdot & \cdot & \cdot & \cdot & 1 \\ t^2p^2 & \cdot & \cdot & \cdot & \cdot & 1 \\ x^2y^2 & xy(x+y) & (x+y)^2 & xy & x+y & 1 \end{vmatrix} = 0.$$

This equation includes extraneous factors, $p-r$, $q-s$, $r-t$, $s-p$, $t-q$. To reduce it by exclusion of these factors and express $\Phi(x, y)$

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$=F(x, y)/\pi(p-r)$, it is most expeditious to use the two conics already mentioned

$$(2)\qquad \begin{aligned} X_1:X_2:X_3 &= x_1^2:x_1x_2:x_2^2,\\ U_1:U_2:U_3 &= x_2y_2: -(x_1y_2+x_2y_1):x_1y_1 \end{aligned}$$

and the pascalian equation for the line-conic. This is then reduced by replacing each 3-rowed determinant by three of two rows each. For example,

$$(3)\qquad \begin{vmatrix} p^2 & p & 1\\ q^2 & q & 1\\ r^2 & r & 1 \end{vmatrix} \equiv -(p-q)(q-r)(r-p).$$

Then the binary identities allow reduction of $F(x, y)$ to any desired standard form such as the following:

$$(4)\qquad \begin{aligned} 0 = {} & 2\cdot(pr)(ps)(rs)\cdot(qx)(qy)\cdot(tx)(ty)\\ & + (qr)(st)\cdot(px)(py)\left\{\begin{array}{l} (qs)[(rx)(ty)+(ry)(tx)]\\ +(rt)[(qx)(sy)+(qy)(sx)] \end{array}\right\}. \end{aligned}$$

One might begin with this standard form of equation and verify by inspection the fact that it is satisfied by each of the five pairs of values (x, y) prescribed. From this explicit equation it can be seen as readily as from the geometric picture that the following is true:

If p, q, r, s, t are real parameters arranged in order of magnitude, then, for example, the passage of x from p to q proceeds pari passu with that of y from q to r.

This simplifies the argument, but eventually the order of arrangement can be arbitrary without impairment of the conclusions.

The correspondence (4) contains at least one cyclic set of five. How is it to be seen that its chains or sequences are all cyclic of period five? For this purpose we attempt to deviate from the initial values p, q, r, s, t, which determine the correspondence, to a neighboring set $p+p', q+q', \cdots, t+t'$, which shall determine the same correspondence, by the *same formal equation*. Since (4) can be regarded as the equation of a line-conic, it is only necessary to make the derived conic intersect it in five points, when the two will necessarily coincide completely. But the pairs (p, q), (q, r), $\cdots$ give five convenient points for comparison. Take the form (1), and vary $p, q, \cdots$ so little that none of the extraneous factors shall vanish. The increment of first order is to yield approximately the equation in (x, y) of our second line-conic. It is

$$0 = p'\frac{\partial F}{\partial p} + q'\frac{\partial F}{\partial q} + \cdots + t'\frac{\partial F}{\partial t} \cdot \tag{5}$$

Examine now $\partial F/\partial p$ for values $x=q$, $y=r$. It vanishes on account of the determinant structure; so also for values (r, s) and for values (s, t), but not for (t, p) nor for (p, q). As further, the increment of the extraneous factor $\prod(p-r)$ (cyclic) is not zero, we have also for the essential factor $\Phi(x, y)$:

The derivative $\partial\Phi(x, y)/\partial p$ becomes zero for three pairs of values (x, y), namely (q, r), (r, s), (s, t); and each of the other derivatives in cyclic order gives in line-coordinates xy, $(x+y)$, 1 the equation of a conic through three of the five points of the given fundamental cycle.

There are therefore five two-term linear equations in $p', q', \cdots, t'$ to be satisfied, namely,

$$\begin{aligned} p'\frac{\partial\phi(p, q)}{\partial p} + q'\frac{\partial\phi(p, q)}{\partial q} &= 0, \\ q'\frac{\partial\phi(q, r)}{\partial q} + r'\frac{\partial\phi(q, r)}{\partial r} &= 0, \\ \cdots\cdots\cdots\cdots\cdots & . \end{aligned} \tag{6}$$

In these, the notation intended is

$$\frac{\partial\phi(p, q)}{\partial p} = \left[\frac{\partial}{\partial p}\Phi(x, y)\right]_{x=p,y=q}.$$

Is the determinant of these equations identically zero?

Recur to formula (4); differentiate, and reduce. Our equations become the following:

$$-(p'p)(pq)(pr)(qs)\cdot(qr)(st)(tq) + (q'q)(qp)(tq)(ps)\cdot(pr)(rs)(tp) = 0,$$

or

$$(p'p)\cdot(qr)(qs)(st) + (q'q)(rs)(ps)(tp) = 0,$$

and the four derived by cyclic permutation. The determinant of these five linear equations for (pp'), (qq'), $\cdots$ has two terms only, and they are identical but of opposite sign.

Hence the equations are consistent and determine the ratios of the five differentials $(p'p)$, $(q'q)$, $(r'r)$, $(s's)$, $(t't)$, uniquely.

In this way we see that there is an infinitesimal variation in the quintette p, q, r, s, t which leaves invariant the equation $\Phi(x, y)=0$ of the (2, 2) correspondence. As each consecutive set is uniquely determined by the variation of one point, the totality of such quintettes is a linear involution. This is a particular instance of the classical theorem (for example, that of Hurwitz) on closed sets in a (2, 2) correspondence; also of Coble's more general theorem on closed sets in any binary (m, n) correspondence.

The formula of the foregoing problem gives a model which serves, with slight modification, in a problem essentially more intricate. Coble has pointed out that since there exists a (3, 3) correspondence with a closed cycle or period of twice seven elements, there must exist also a (6, 6) correspondence with a closed cycle of seven elements, wherein accordingly each corresponds to all the rest. To construct its explicit expression, let the seven elements be $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \theta$. Arrange them cyclically in seven triads, as a Δ_7: $\alpha\beta\delta$, $\beta\gamma\epsilon$, $\cdots$, $\theta\alpha\gamma$. Form binary cubics in (u) such as $(u\alpha)(u\beta)(u\gamma)$, whose coefficients shall be taken as quaternary coordinates, $\xi_1' : \xi_2' : \xi_3' : \xi_4'$. Quadric products of these give ten column heads, and the remaining six triads fill the array to seven rows. Fill the last three rows similarly from triads πxy, ρxy, σxy, and close the determinant, then divide by the obvious factor $(\pi\rho)(\pi\sigma)(\rho\sigma)$. Call the quotient $D_{10}/(\pi\rho)(\pi\sigma)(\rho\sigma)=\Phi(x, y)$.

The property of this double sextic Φ is that it vanishes for every pair of values $x=\alpha$, $y=\beta$, $\cdots$ and is symmetric in x and y.

The latter is visibly true. As to the former, consider D_{10}. It contains α, β in each of the last three rows; and, by the definition of the triad system, once in some earlier row. Thus since the four quantics $(u\pi)^2(u\alpha)^2(u\beta)^2$, $(u\rho)^2(u\alpha)^2(u\beta)^2$, $(u\sigma)^2(u\alpha)^2(u\beta)^2$, and $(u\gamma)^2(u\alpha)^2(u\beta)^2$ are linearly dependent, so also are the quadrics whose coefficients form four rows in D_{10}. Therefore D_{10} vanishes, and the $\Phi(\alpha, \beta)=0$.

The equation $\Phi(x, y)=0$ is satisfied whenever a plane through the two points x, y of a gauche cubic curve (on which seven points represent the parameters α, β, γ, $\cdots$) osculates the second gauche cubic determined by six planes of the Δ_7.

For such a plane through x, y will have the square of its equation linearly compounded from those given by the last three rows of D_{10}; and, on the other hand, as it osculates the second cubic envelope, it must be linearly compounded from the first seven. Hence there is a linear identity among the ten quadrics corresponding to the rows of the D_{10}; whence follows the above proposition.

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