ON THE nTH DERIVATIVE OF $f(x)$*

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Let $y_1, y_2, y_3, \cdots$ be defined recursively as follows: y_1 is the logarithmic derivative of a function $y=f(x)$, and $y_\nu = D_x y_{\nu-1}$, $(\nu=2, 3, 4, \cdots)$. Then the successive derivatives $y', y'', y''', \cdots$ of y with respect to x are polynomials in y and the y_ν. In fact, $y'=yy_1$, $y''=y(y_2+y_1^2)$, $y'''=y(y_3+3y_1y_2+y_1^3)$, and

$$y^{(n)} = y \sum A^{(n)}_{\nu_1\nu_2\cdots\nu_n} y_1^{\nu_1} y_2^{\nu_2} \cdots y_n^{\nu_n}, \tag{1}$$

where $A^{(n)}_{\nu_1\nu_2\cdots\nu_n}$ is a positive integer and the summation is taken for all non-negative integral solutions $\nu_1, \nu_2, \nu_3, \cdots, \nu_n$ of the equation

$$\nu_1 + 2\nu_2 + 3\nu_3 + \cdots + n\nu_n = n. \tag{2}$$

This statement may readily be proved by mathematical induction. The principal object of the present note is to prove the following theorem:

THEOREM. *The integer $A^{(n)}_{\nu_1\nu_2\cdots\nu_n}$ in (1) is equal to the number of ways that n different objects can be placed in compartments, one in each of ν_1 compartments, two in each of ν_2 compartments, three in each of ν_3 compartments, $\cdots$, without regard to the order of arrangement of the compartments.*

1. Generalized binomial coefficients. Let k, m, n, $(kn \leqq m)$, be positive integers, and denote by $C^{(k)}_{m,n}$ the number of ways that kn objects can be selected from m objects and placed in n compartments, k in each compartment, where no account is taken of the order of arrangement of the compartments. Thus $C^{(1)}_{m,n}$ is the binomial coefficient $C_{m,n}=m!/[n!(m-n)!]$. We have

$$n!\cdot C^{(k)}_{m,n} = C_{m,kn}\cdot(C_{kn,k}\cdot C_{k(n-1),k}\cdot \cdots \cdot C_{k,k}),$$

or

$$C^{(k)}_{m,n} = m!/[n!(m-kn)!(k!)^n]. \tag{3}$$

This has meaning if $m \geqq kn$. For special 0 values of the indices we shall consider $C^{(k)}_{m,n}$ to be defined by (3) by taking $0!=1$. Thus if $k\geqq 0$, $m \geqq 0$, we have $C^{(k)}_{m,0}=1$.

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If (2) holds, it will be seen that

$$(4)\qquad C^{(1)}_{k_1,\nu_1}\cdot C^{(2)}_{k_2,\nu_2}\cdot\ \cdots\ \cdot C^{(n)}_{k_n,\nu_n} = \frac{n!}{\nu_1!\nu_2!\cdots\nu_n!(1!)^{\nu_1}(2!)^{\nu_2}\cdots(n!)^{\nu_n}},$$

where $k_1=n$, $k_2=n-\nu_1$, $k_3=n-\nu_1-2\nu_2$, $\cdots$. We are to prove that this is the value of $A^{(n)}_{\nu_1\nu_2\cdots\nu_n}$ in (1). For the proof we need the following identities which will be seen to hold for all values of m, n, k for which the symbols involved have been defined:

$$(5)\qquad C^{(k+1)}_{m,n} = C^{(k+1)}_{m-1,n} + C^{(k)}_{m-1,1}\cdot C^{(k+1)}_{m-k-1,n-1},$$

$$(6)\qquad (n+1)\cdot C^{(k)}_{m,n+1} = C^{(k)}_{m,n}\cdot C^{(k)}_{m-kn,1}.$$

Let it be remarked in passing that if $P_{n,k}=C^{(k)}_{n,0}+C^{(k)}_{n,1}x+C^{(k)}_{n,2}x^2+\cdots$, then from (5) it follows that $P_{n,k}=P_{n-1,k}+C_{n-1,k-1}xP_{n-k,k}$. Also $P'_{n,k}=C_{n,k}P_{n-k,k}$.

2. **Derivation of the formula for $A^{(n)}_{\nu_1\nu_2\cdots\nu_n}$.** Denote the sum in (1) by S_n, and write S_n as a polynomial in y_1, $S_n=\sum_{\nu=0}^{n}S^{(1)}_{n,\nu}y_1^{\nu}$, where $S^{(1)}_{n,\nu}$ is independent of y_1. We begin by showing that

$$(7)\qquad S^{(1)}_{i,j} = C^{(1)}_{i,j}S^{(1)}_{i-j,0}, \qquad 0\leqq j\leqq i.$$

We use induction on the subscript difference $i-j=k$. From the relation $yS_n=D_x[yS_{n-1}]$ it follows that

$$(8)\qquad S^{(1)}_{n,\nu} = S^{(1)}_{n-1,\nu-1} + (1+\nu)y_2 S^{(1)}_{n-1,\nu+1} + D_x S^{(1)}_{n-1,\nu}, \qquad \nu = 0, 1, 2, \cdots, n,$$

with the agreement that $S_{i,j}=0$ if $j<0$ or $j>i$. Assuming that (7) holds for $k<q$ we shall prove that it holds for $k=q$. Accordingly, we choose n, ν, $(0\leqq\nu<n)$, in (8) so that $n-\nu=q$. Then, by our assumption, (8) may be written in the form

$$(9)\qquad S^{(1)}_{n,\nu} = \begin{cases} S^{(1)}_{n-1,\nu-1} + (1+\nu)y_2 C^{(1)}_{n-1,\nu+1}S^{(1)}_{n-\nu-2,0} + C^{(1)}_{n-1,\nu}D_x S^{(1)}_{n-\nu-1,0}, & \text{if } q>1;\\ S^{(1)}_{n-1,\nu-1} + C^{(1)}_{n-1,\nu}D_x S^{(1)}_{n-\nu-1,0}, & \text{if } q=1.\end{cases}$$

Replace n by $n-\nu$, and ν by 0 in (8), and eliminate $D_xS^{(1)}_{n-\nu-1,0}$ in (9). The result, by (6), is

$$(10)\qquad S^{(1)}_{n,\nu} = S^{(1)}_{n-1,\nu-1} + C^{(1)}_{n-1,\nu}S^{(1)}_{n-\nu,0}.$$

Hence

$$S_{n,\nu}^{(1)} = \sum_{i=0}^{\nu} [S_{n-i,\nu-i}^{(1)} - S_{n-i-1,\nu-i-1}^{(1)}] = \left[\sum_{i=0}^{\nu} C_{n-i-1,\nu-i}^{(1)}\right] S_{n-\nu,0}^{(1)};$$

or, by (5) with $k=0$, $S_{n,\nu}^{(1)} = C_{n,\nu}^{(1)} S_{n-\nu,0}$, as was to be proved.

We next put

$$S_{m,0}^{(p-1)} = \sum_{\nu=0}^{[m/p]} S_{m,\nu}^{(p)} y_p^{\nu}, \qquad p = 2, 3, 4, \cdots .$$

Then the formulas

$$S_{m,\nu}^{(p)} = C_{m,\nu}^{(p)} S_{m-p\nu,0}^{(p)}, \tag{11}$$

$$S_{m,\nu}^{(p)} = C_{m-1,1}^{(p-1)} S_{m-p,\nu-1}^{(p)} + (1+\nu) y_{p+1} S_{m-1,\nu+1}^{(p)} + D_x S_{m-1,\nu}^{(p)} \tag{12}$$

hold for $p=1$. Assuming that they hold for $p<k$, $k>1$, we may then prove them for $p=k$. To do this, put $\nu=0$ and $p=k-1$ in (12), and equate coefficients of like powers of y_k. The result is the equation (12) with $p=k$. Thus (12) is true when $p=k$, and in particular $S_{k\nu,\nu}^{(k)} = C_{k\nu-1,1}^{(k-1)} S_{k(\nu-1),\nu-1}^{(k)}$. Hence by (5) we find that $S_{k\nu,\nu}^{(k)} = C_{k\nu,\nu}^{(k)} S_{0,0}^{(k)}$; so that (11) holds for $p=k$ provided $m-k\nu=0$. The proof of (11) for $p=k$ may now be carried out along the lines of the proof of (7), with induction, in this case, on the difference $m-k\nu$.

After (11) has been proved it follows at once by (4) that

$$A_{\nu_1\nu_2\cdots\nu_n}^{(n)} = \frac{n!}{\nu_1!\nu_2!\cdots\nu_n!(1!)^{\nu_1}(2!)^{\nu_2}\cdots(n!)^{\nu_n}}. \tag{13}$$

3. **Application.** In conclusion I shall give examples to illustrate the application of the foregoing result.

EXAMPLE 1. Let $a_{n,k}$ denote the number of ways that n different objects can be distributed among k compartments, where no account is taken of the order of arrangement of the compartments, and at least one object is placed in each compartment. Then elementary considerations will show that $a_{n,k} = ka_{n-1,k} + a_{n-1,k-1}$. Put

$$y = e^{te^x} = e^t \sum_{\nu=0}^{\infty} L_\nu(t) x^\nu/\nu!.$$

Then by (1) we find that $L_n(t) = a_{n,1}t + a_{n,2}t^2 + \cdots + a_{n,n}t^n$.

Put $g_k(y) = \sum_{\nu=0}^{\infty} a_{k+\nu,k} yt$. It follows that $g_1(y) = 1/(1-y)$, $g_k(y) = g_{k-1}(y)/(1-ky)$, $(k=2, 3, 4, \cdots)$, or $g_k(y) = 1/[(1-y)(1-2y) \cdots (1-ky)]$. Hence $L_n(1)$, the number of ways that n different objects can be distributed among n or fewer compartments, is the coefficient of y^n in the power series $P(y)$ for the function

$$\frac{y}{(1-y)}+\frac{y^2}{(1-y)(1-2y)}+\cdots+\frac{y^m}{(1-y)(1-2y)\cdots(1-my)},$$

where $m \geqq n$. The number $L_n(1)$ is also the coefficient of $x^n/n!$ in the power series for the function $e^{(e^x-1)}$.

The $a_{n,k}$ are given explicitly by the formula

$$a_{n,k}=\frac{(-1)^{k+1}}{k!}\sum_{\nu=0}^{k} C_{k,\nu}(-1)^{\nu+1}\nu^n.$$

EXAMPLE 2. Put $y=(1+x)^{-t}$ in (1) and then set $x=0$. There results this identity:

$$\begin{aligned}\frac{t(t+1)(t+2)\cdots(t+n-1)}{n!}\\ =\sum\frac{t^{\nu_1+\nu_2+\cdots+\nu_n}}{(1\cdot 2\cdot\ \cdots\ \cdot\nu_1)(2\cdot 4\cdot\ \cdots\ \cdot 2\nu_2)\cdots(n\cdot 2n\cdot\ \cdots\ \cdot n\nu_n)},\end{aligned}\tag{14}$$

where the summation is taken as in (1). On putting $t=1$ in (14) we obtain the following theorem:*

THEOREM. *Form a partition of n by taking at most one integer from each of the progressions* 1, 2, 3, $\cdots$; 2, 4, 6, $\cdots$; 3, 6, 9, $\cdots$; $\cdots$. *Multiply together the terms of each progression up to and including the integer chosen. Let the products so formed be a, b, c, $\cdots$. Then* $\sum[1/(a\cdot b\cdot c\cdot\ \cdots\ )]=1$, *where the sum is taken for all such partitions of n.*

EXAMPLE 3. If we differentiate the members of (14) with respect to t and then set $t=1$, we get the formula

$$\begin{aligned}1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\\ =\sum\frac{(\nu_1+\nu_2+\cdots+\nu_n)}{(1\cdot 2\cdot\ \cdots\ \cdot\nu_1)(2\cdot 4\cdot\ \cdots\ \cdot 2\nu_2)\cdots(n\cdot 2n\cdot\ \cdots\ \cdot n\nu_n)}.\end{aligned}$$

This may likewise be interpreted as a theorem on partitions of n.

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* Jacobi, *Zur combinatorischen Analysis*, Crelle's Journal, vol. 22 (1841), pp. 372–374.