CARTAN ON PROJECTIVELY CONNECTED SPACES

Leçons sur la Théorie des Espaces à Connexion Projective. By Élie Cartan. Edited by P. Vincensini. (Cahiers Scientifiques, publiés sous la direction de M. Gaston Julia, no. 17.) Paris, Gauthier-Villars, 1937. vi+308 pp.

In a recent monograph, La Méthode de Repère mobile, · · · (Actualités Scientifiques et Industrielles, no. 194; reviewed in this Bulletin, November, 1935) E. Cartan presented a splendid outline of his general method of approach toward all branches of differential geometry. In a word, it consists of a far-reaching generalization of the familiar moving trihedral with assistance from the theory of groups. The analytical formulation employs the exterior differential calculus, a discipline extensively used by Cartan since the turn of the century. And there is frequent recourse to the theory of Pfaffian systems. Undoubtedly his unusual analytical machinery is, to many, a source of difficulty. Most differential geometers use, instead, Ricci's tensor calculus and theorems on total differential equations stemming from Christoffel. But every disciple of Ricci knows the profit which lies in the study of Cartan.

In the above monograph the author devoted one paragraph to showing how projective differential geometry fits into his general scheme. In the work here under review this paragraph is presented to us in more satisfactory form as a book of three hundred odd pages.

The book has two principal divisions, the first devoted to classical projective differential geometry, the second, to the geometry of projectively connected spaces. The first chapter is concerned with the projective line, both real and complex. Since from the intrinsic standpoint any two one-dimensional loci are locally equivalent, the purely differential-geometric discussion of the real line is trivial. But by introducing motion, Cartan finds a kinematical theory which is a good introduction to his method of moving reference systems, systems which in this book are always simplexes defining homogeneous projective coordinates. A pleasing detail is his interpretation of the Schwarzian derivative as a projective acceleration. The material on the complex line shows how moving reference systems can be used in the complex domain. This subject he has treated at length in a previous work.

In the second chapter the elements of the theory of plane curves is first developed by Wilczynski's method. Here the author adds an exciting definition of the projective arc. Then, returning to his own method of moving reference systems, he redevelops the theory twice and in some detail, first with aid from the method of reduced equations, and finally directly. In the direct method geometrical intuition is ignored and the invariant theory is developed purely analytically. This attack makes for difficulty but has the advantage of being applicable in general situations where intuition fails to suggest shortcuts. In the present case it leads to an intrinsic reference simplex of the sixth order associated with each point of a curve, in terms of which the projective analogues of the Frenet equations take their simplest form. The chapter ends with a derivation of the so-called structure equations of the projective group and a simplification of the direct method by their use. At one point in this chapter the author makes the surprising statement that curves of constant projective curvature (for example, $y=x^3$) have no inflection points.

The third chapter carries the first division of the work to its conclusion with a discussion of surfaces in three-space. The method of reduced equations leads to the differential forms of Fubini and the projective line element. The quadrics and lines of Darboux and the quadric of Lie receive an unusually clear analytic and geometric characterization. Strangely enough the usual interpretation of the linear element as a cross ratio is not mentioned. The author then proceeds to apply his direct method of moving reference systems. Starting with general coordinate simplexes associated with each point of the surface, restrictions of successively higher differential order are applied until a unique system of the fourth order is invariantively determined at each point. This development leads to the six fundamental invariants of a surface. The chapter ends with an interesting section on projectively applicable surfaces, in particular, on the order of contact obtainable at homologous points.

The second division of the work opens by crossing the divide between projective space and a space with a projective connection. Since most investigators seem content to stay on one side or the other, projective differential geometry has been strongly divided into two schools. The author, however, belongs to both, and with fine skill carries us from the one to the other without unpleasant jolts.

The projective connection of an n-space with general coordinates u^i consists of $n(n+2)$ Pfaffians $\omega_\beta^\alpha = \prod_{\beta i}^\alpha (u) du^i$, with $\omega_0^0=0$ and $\alpha, \beta = 0, 1, \cdots, n$, which define a transformation $dA_\alpha = \omega_\alpha^\beta A_\beta$ from a local reference simplex, of vertices A_α, at u^i to a simplex, of vertices $A_\alpha + dA_\alpha$, at $u^i + du^i$. The simplex at each point u^i is regarded as lying in a projective n-space associated with this point, the points A_0 and u^i being identified. It is shown that the connection determines an invariantive application of the second order between a neighborhood of any point u^i of the base space and a neighborhood of A_0 in the space associated with u^i. Abstractly, therefore, the associated spaces may be regarded as tangent spaces. When the torsion is zero the above application may be made to the third order. Thus Cartan uses the assigned connection to batten down the associated spaces. A slightly more general procedure followed by some writers is to batten down the associated spaces before assigning the connection. The author considers how the ω_β^α change when the local reference simplexes are changed, and arrives at a certain "natural" reference system to be associated with a given system in the base space by normalizing the ω_β^α in an invariant way.

In Chapter 2 the tensor of curvature and torsion is derived in the usual manner by displacing a tangent space about an infinitesimal parallelogram in the base space. The equations obtained by setting this tensor equal to zero are just the equations of structure of the projective group. Hence the vanishing of this tensor is necessary and sufficient for the generalized space to be locally projective. This leads in the following chapter to a discussion of projective tensor calculus and there results a most interesting and unusual development of this discipline. It is an application of the general theory presented by the author in his address to the Society at the Harvard Tercentenary, published in the Annals of Mathematics (January, 1937). His projective tensors and their algebra turn out to be essentially the same as Veblen's. Different, however, is his covariant differentiation, defined geometrically and applicable to any field quantity. But even when applied to a tensor it does not give rise in general to another tensor. As an application of this calculus the author proves that a space with a projective connection is completely characterized by the curvature-torsion tensor and its successive covariant derivatives. The proof rests on the use of normal coordinates (usual definition) and particular reference simplexes invariantively associated with them. At the origin of normal coordinates Cartan's covariant derivatives reduce to ordinary derivatives.

After a short chapter devoted to the projective generalization of the identities of

Bianchi and their geometrical interpretation, there follows in Chapter 5 a geometrization of certain second order differential equations. Limiting himself first to two dimensions the author shows that with each equation of the form

$$\frac{d^2v}{du^2} = A\left(\frac{dv}{du}\right)^3 + 3B\left(\frac{dv}{du}\right)^2 + 3C\frac{dv}{du} + D,$$

with $A, \cdots, D$ functions of u, v, there can be invariantively associated a certain "normal" projectively connected space having this equation for the equation of its geodesics. The generalization of this result to n dimensions leads to a connection identical with the one associated with a system of paths by T. Y. Thomas. As an illustration of the application of projective methods to metric geometry there is a discussion of geodesic representation of surfaces in euclidean three-space using the projective connection associated with the geodesics.

Chapter 6 is devoted to the study of a few typical problems concerning surfaces in projectively connected three-space. For example, it is proved that the vanishing of a certain tensor, explicitly given, is a necessary and sufficient condition that every surface of the space have reciprocally conjugate tangents. Considerable attention is given to the problem of generalizing the tangents of Darboux, with the conclusion that several alternative definitions are possible leading to distinct configurations.

The final chapter approaches projectively connected spaces through their holonomic group, a notion which has been one of the author's most fruitful contributions to differential geometry. This approach, unfamiliar to many, is treated with unusual clarity. General directions are given for attacking the two fundamental problems: (1) given a projectively connected space, to find its holonomic group, and (2) given a group, to determine the possible spaces having it for holonomic group. Problem 2 is solved for several important groups. Last comes a determination of all holonomic groups of "normal" projective two-spaces. There is a bibliography which seems short and rather haphazard.

Of all Cartan's books this is one of the most clearly written but, like the others, it can be fully appreciated only by those having some previous acquaintance with its field. The author works through his subject informally, enriching it with his unified point of view and his unsurpassable geometric insight, finding new approaches leading to a better understanding, and giving here and there a new result for good measure. This is probably the first major publication in book form to have classical and generalized projective differential geometry fraternizing within its covers. And the author was a pioneer in both fields.

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