we have

$$\lim_{k\to\infty} \delta_i^{(k)} = \prod_j^{(i)} d_{i,j} \prod_j D_{i,j}.$$

Now

$$\left| \delta_i^{(k)} - \delta \right| = \delta \left| 1 - \frac{\prod_j^{(i)} D_{i,j}^{(k)}}{\prod_j^{(i)} D_{i,j}^{(k-1)}} \right|.$$

But the quantity on the right approaches zero, so that $\delta_i^{(k)} \to \delta$ as $k \to \infty$. We thus have (1), and the theorem is proved.

INSTITUTE FOR ADVANCED STUDY

AN INVOLUTORIAL LINE TRANSFORMATION IN S_4

BY C. R. WYLIE, JR.

1. *Introduction.* It is a well known fact that all planes which meet four general lines of S_4* are met by a fifth line. The remarkable configuration determined by five such "associated lines" is discussed in a number of places in the literature.† In the present paper an involutorial line transformation suggested by the figure of five associated lines is discussed, both as a line involution in S_4, and as a point involution on a certain V_6^5 in S_9. In §§2–6 the involution is treated at some length by purely synthetic methods. The final section (§7) contains a brief analytic treatment, including the equations of the involution, and the equations of the invariant and singular elements. The involu-

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* We shall use the conventional symbol S_m to indicate a linear space of dimension m. A variety of order r and of dimension m we shall designate by the symbol V_m^r.

† Welchman, W. G., *Plane congruences of the second order in space of four dimensions and fifth incidence theorems*, Proceedings of the Cambridge Philosophical Society, vol. 28 (1931–1932), pp. 275–284.

Baker, H. F., *On a proof of the theorem of a double six of lines by projection from four dimensions*, Proceedings of the Cambridge Philosophical Society, vol. 20 (1920–1921), pp. 133–144.

Baker, H. F., *Principles of Geometry*, Cambridge University Press, 1925, vol. IV, Chapter V, tion appears to be an important one in that most of the loci connected with it are of considerable interest in the geometry of hyperspace.

2. *Representation of Five Associated Lines in S_9.* Stephanos has shown* that five associated lines have their ten Grassmann coordinates linearly related. If we regard the ten Grassmann coordinates of the lines of S_4 as point coordinates in S_9,† and recall that the five quadratic identities existing among these coordinates define a V_6^5 in S_9‡ whose points are in 1:1 correspondence with the lines of S_4, this result may be interpreted as follows: The images in S_9 of five associated lines of S_4 are the five points in which a general S_3 meets V_6^5. In fact, from this as a definition of five associated lines the incidence property mentioned above follows at once. For consider four general lines, L_1, L_2, L_3, L_4, of S_4, and any plane, P, meeting each of these lines. On V_6^5 we have the four independent points l_1, l_2, l_3, l_4, which are the images of the four lines L_1, L_2, L_3, L_4.§ Now the S_8 which intersects V_6^5 in the V_5^5 representing the special complex of lines which meet the plane P contains l_1, l_2, l_3, l_4. It therefore contains l_5, the fifth point in which the S_3 determined by l_1, l_2, l_3, l_4 intersects V_6^5. Hence l_5 is also the image of a line L_5, which meets the plane P.

3. *Definition of the Involution.* Let there be given three general lines, L_1, L_2, L_3, of S_4 and let the transform of any line L be the line L' which forms with L_1, L_2, L_3, and L a set of five associated lines. On V_6^5 this becomes a point involution defined as follows: Let there be given three points, l_1, l_2, l_3, on V_6^5 such that the plane, π, which they determine meets V_6^5 only in these three points. Then the transform of any point, l, is the fifth point, l', in which the S_3 determined by l_1, l_2, l_3, and l meets V_6^5.

* Stephanos, C., *Sur une configuration remarquable de cercles dans l'espace*, Comptes Rendus, vol. 93 (1881), p. 578.

† Todd, J. A., *The locus representing the lines of four dimensional space and its application to linear complexes in four dimensions*, Proceedings of the London Mathematical Society, (2), vol. 30 (1929–1930), pp. 513–550.

‡ See §7 for the analytic definition of the Grassmann coordinates, and of the five quadratic identities.

§ Throughout this paper we shall represent configurations in S_4 by capital letters, and their images in S_9 by small letters.

To determine the order of the involution consider a general pencil of lines, R, of S_4. The image of this pencil on V_6^5 is a line, r. The ∞^1 spaces S_3 determined by π and the respective points of r all lie in the S_4 determined by π and r. This S_4 meets V_6^5 in a curve of order five. The line r is a point of this curve of intersection. The other component is a rational normal quartic curve, passing through the three points l_1, l_2, l_3 in π. This quartic curve is the transform of r in the involution on V_6^5, hence the involution in S_4 is of order four, transforming a pencil of lines into a ruled quartic surface, three of whose generators are the fundamental lines L_1, L_2, L_3.

4. *Invariant Elements*. There are ∞^4 varieties V_4^2 on V_6^5, these being the representations in S_9 of the lines of the ∞^4 spaces S_3 in S_4.* Each of these V_4^2's lies in an S_5. One and only one of these S_5's passes through a general point of S_9, while ∞^2 of them pass through each point of V_6^5. In particular the ∞^2 spaces S_5 corresponding to the points of π cut V_6^5 in V_4^2's each of which contains the point, t, which is the representation in S_9 of the common transversal, T, of the lines L_1, L_2, L_3, in S_4. These V_4^2's are thus representations of the lines of the ∞^2 spaces S_3 through T in S_4.

Consider now a point, l, and its transform, l', in the involution on V_6^5, and consider further the S_5 corresponding to the point, q, in which the line joining l and l' intersects π. Any other point, m, on the V_4^2 cut from V_6^5 by this S_5 must have for its transform, m', another point of the same V_4^2, namely the second point in which the line joining q and m intersects the V_4^2. Thus the involution on V_6^5 leaves invariant as a whole, though not point by point, the ∞^2 varieties V_4^2 corresponding to the points of π. This implies that in S_4 the transform of any line L lies in the S_3 determined by L and the transversal, T, of the three lines L_1, L_2, L_3.

The invariant points in the involution on V_6^5 are the points of contact of the tangents to V_6^5 which meet π. A triple infinity of these tangents pass through each point, q, of π. They are in fact the ∞^3 tangents to the V_4^2 corresponding to the point q; and the locus of their points of contact is the V_3^2 cut from the V_4^2 by the polar S_4 of q as to the V_4^2 (the polar S_4 being deter-

* Todd, loc. cit.

mined in the S_5 containing q and the corresponding V_4^2). Thus in S_4 the invariant lines of the involution form a linear complex in each S_3 through T.

On V_6^5 there are ∞^3 planes with the property that they contain three non-concurrent lines which determine with the respective points l_1, l_2, l_3 planes lying entirely on V_6^5. (These planes are the representations in S_9 of the fields of lines lying in planes which meet L_1, L_2, L_3.) Each of the three lines in such a plane, σ, is singular in the involution, its image being the plane which joins it to the corresponding point, l_i, plus two lines joining l_i to the other two points, l_j and l_k. The image of σ itself is the residual V_2^4 cut from V_6^5 by the S_5 determined by π and σ. This V_2^4 is composite, consisting of the three planes which are the images of the three singular lines in σ, and a fourth plane, σ'. Now σ' must meet in a line each of the other planes composing V_2^4, since it lies on V_6^5 and meets in a line the S_4 determined by π and each of these planes. But this requires either that σ' coincide with σ, or that σ and the planes composing V_2^4 lie in an S_4 instead of in an S_5. The latter is impossible, hence σ and σ' coincide, and the non-singular points of σ are invariant. Conversely, every invariant point on V_6^5 lies in a plane σ, namely the plane common to the three V_4^2's determined by the invariant point and l_1, l_2, l_3, respectively.

Thus in S_4 the invariant lines are the ∞^5 lines which lie in planes meeting L_1, L_2, L_3. Hence in a general S_3 of S_4 the invariant lines form a tetrahedral complex, whose fundamental tetrahedron is determined by the four points in which the S_3 intersects L_1, L_2, L_3, and T.* In an S_3 through T this quadratic complex breaks up into a general linear complex, as we have already noted, and the special linear complex of lines meeting T. the latter complex being singular and not properly invariant.

5. *Singular Elements*. The singular elements in the involution on V_6^5, aside from the three fundamental points l_1, l_2, l_3, are those points which determine with π an S_3 meeting V_6^5 in a curve instead of in five points. They are of three general types: Those points which lie on lines of V_6^5 passing through l_i, those

* The general tetrahedral complex can be formed by sectioning with an S_3 the system of all planes meeting three general lines of S_4. Cf. Baker, *Principles of Geometry*, Vol. IV, p. 32.

points which lie on conics of V_6^5 passing through l_i and l_j, and those points which lie on cubics of V_6^5 passing through l_i, l_j, l_k. The last class of points is also the class of points lying on lines of V_6^5 which pass through t. This is evident when we consider the S_4 determined by t and an S_3 containing a cubic curve through l_1, l_2, l_3. Such an S_4 contains not only the cubic curve but also the lines joining t to l_1, l_2, l_3. Since these lines lie on V_6^5, the S_4 thus meets V_6^5 in a curve of order greater than five, and hence has a surface in common with V_6^5. This surface must be a cubic cone with vertex at t, and the given cubic as directrix curve. Hence all points on any cubic through l_1, l_2, l_3 lie on lines through t.

The three classes of singular points correspond respectively to the following classes of singular lines in S_4: Lines meeting L_i, lines lying in the S_3 determined by L_i and L_j, and lines meeting T. The images of lines belonging to one or more of these classes may be described as follows: (The symbol $\sim$ is used to mean "is transformed into.")

1. A general line, L, meeting $T \sim$ the cubic regulus containing L_1, L_2, L_3, and L as generators, and having T as directrix.

2. A general line, L, in the S_3 determined by L_i and $L_j \sim$ the quadratic regulus determined by L_i, L_j, and L.

3. A general line, L, meeting $L_i \sim$ the pencil determined by L_i and L.

4. A general line, L, through the intersection of T and $L_i \sim$ the pencil determined by T and L_i, together with the quadratic regulus determined by L_j, L_k, and the line of the pencil which lies in the S_3 determined by L_j and L_k.

4.1. A general line, L, meeting T and lying in the S_3 determined by L_i and $L_j \sim$ the quadratic regulus determined by L_i, L_j, and L, together with the pencil determined by L_k and the generator of the regulus which passes through the intersection of T and L_k.

5. A line, L, in the S_3 determined by L_i and L_j passing through the intersection of T and $L_i \sim$ the pencil determined by L_i and L, the pencil determined by L_j and the line of the first pencil which meets L_j, and the pencil determined by L_k and the line of the second pencil which meets L_k.

5.1. A general line L, in the plane of T and $L_i \sim$ the pencil determined by L_i and L, the pencil determined by L_j and the line of the first pencil which meets L_j, and the pencil determined by L_k and the line of the first pencil which meets L_k.

6. A general line, L, meeting L_i and $L_j \sim$ the pencil determined by L_i and L, together with the pencil determined by L_j and L.

6.1. A general line, L, meeting L_i and lying in the S_3 determined by L_i and $L_j \sim$ the pencil determined by L_i and L, together with the pencil determined by L_j and the line of the first pencil which meets L_j.

7. The transversal, T, $\sim$ the three pencils determined respectively by T and L_i, T and L_j, and T and L_k.

8. The fundamental line, L_i, $\sim$ the special linear complex consisting of all lines which meet the plane determined by T and L_i.

None of these but the last needs special comment; we verify it in this fashion: Any point which is an image of l_i in the involution on V_6^5 must be such a point that the S_3 which it determines with π contains one of the tangents to V_6^5 at l_i. There are ∞^5 tangents which can be drawn to V_6^5 at the point l_i. All these points lie in the S_8 determined by π and the tangent S_6 to V_6^5 at l_i, and hence correspond to lines of a special linear complex in S_4. The singular plane of this complex must contain L_i, and must be met by L_j, and L_k. It must therefore be the plane determined by T and L_i.

6. *Images of Linear Systems of Lines.* As we have already noted, every line, L, which meets either L_1, L_2, or L_3, is transformed into a pencil containing L. Although such lines are singular they may thus be regarded as invariant also. Hence L_1, L_2, and L_3 must lie upon all the quadratic hypercones formed by the invariant lines which pass through the respective points of S_4. Moreover L_1, L_2, and L_3 lie in planes of the same family on each of these hypercones.

Consider now the ∞^3 lines through a general point of S_4. On V_6^5 these are represented by the points of an S_3. The transform of this S_3 is the residual V_3^4 in which the S_6 determined by π and the S_3 intersects V_6^5. Now the S_8 whose intersection with V_6^5 represents the special complex of lines meeting *any* plane of the second family (the family not containing L_1, L_2, and L_3) on the hypercone of invariant lines through the given point in S_4, contains l_1, l_2, l_3 and the S_3 which represents the totality of lines through the given point. The S_8 therefore contains the entire S_6 in which the image V_3^4 lies; hence all points of V_3^4 represent lines of S_4 which meet all planes of the second family on the hypercone of invariant lines. Thus V_3^4 must represent the totality of lines lying in the planes of the first family on the hypercone, and this system is then the transform of the ∞^3 lines through a general point of S_4.

Since the double infinity of lines through a general point in a general S_3 in S_4 is included in the totality of all lines through the point, the transform of such a system must be ∞^2 lines lying in planes of the first family on the hypercone of invariant lines through the point. These lines are in fact the lines which lie in planes of the first family and meet the plane determined by the points where the S_3 intersects L_1, L_2, and L_3. For on V_6^5 the image of the given system of lines is a plane, and its transform is the residual V_2^4 in which the S_5 determined by π and this plane meets V_6^5. Now the S_8 which meets V_6^5 in the V_5^5 representing the special complex of lines meeting the plane, P, determined in S_4 by the points where the given S_3 intersects L_1, L_2, and L_3, contains l_1, l_2, l_3, and the plane which represents the given system of lines. It therefore contains the S_5 in which the image V_2^4 lies. Hence all points of this V_2^4 represent lines meeting the plane P. Since the family of lines represented by this V_2^4 contains ∞^1 pencils of lines (one in each plane of the first family on the hypercone), the V_2^4 must be the rational normal ruled V_2^4 of S_5, and not the Veronese surface.

We have already noted that the transform of a general pencil of lines of S_4 is a rational ruled quartic surface containing L_1, L_2, L_3, and two lines of the given pencil, namely the two invariant lines. These two lines are the only generators of the surface which can intersect. This surface, the projection of the rational normal ruled quartic surface of S_5, must also lie on the hypercone of invariant lines through the vertex of the pencil. In fact this hypercone is the only quadratic primal on which the surface can lie.*

To determine the image of a plane field of lines in S_4 consider the S_5 which is determined in S_9 by π and the plane, p, which

* Cf. Baker, *Principles of Geometry*, vol. II, pp. 275, 279.

represents the lines of the given plane field. This S_5 intersects V_6^5 in the plane p, and in a V_2^4, the Veronese surface, which is the transform of p. Now consider the S_8 whose intersection with V_6^5 represents the special linear complex of lines meeting *any* one of the planes determined in S_4 by the points where an S_3 through the plane field intersects L_1, L_2, and L_3. This S_8 contains l_1, l_2, l_3, and the plane p. It therefore contains the S_5 determined by π and p, and hence contains V_2^4. All points of V_2^4 are thus representations of lines of S_4 which meet all the planes determined by the respective triads of points in which the S_3's through the plane field intersect L_1, L_2, and L_3. As a point locus this system of lines is a V_3^3 having a plane of double points.*

7. *Analytic Procedure.* Let the three fundamental lines in S_4 be determined by the respective pairs of points: (10000)(01000), (00100)(00010), (00001)(10100). The common transversal of these lines is the line joining (10000) and (00100). The Grassmann coordinates of these lines, as read from the matrices of the points which determine each line, are

	P_{12}	P_{13}	P_{14}	P_{15}	P_{23}	P_{24}	P_{25}	P_{34}	P_{35}	P_{45}
$L_1 : l_1 :$	1	0	0	0	0	0	0	0	0	0
$L_2 : l_2 :$	0	0	0	0	0	0	0	1	0	0 .
$L_3 : l_3 :$	0	0	0	1	0	0	0	0	1	0
$T : t :$	0	1	0	0	0	0	0	0	0	0

The quadratic identities which define V_6^5 are

$$P_{23}P_{45} - P_{24}P_{35} + P_{25}P_{34} = 0,$$
$$P_{13}P_{45} - P_{14}P_{35} + P_{15}P_{34} = 0,$$
$$P_{12}P_{45} - P_{14}P_{25} + P_{15}P_{24} = 0,$$
$$P_{12}P_{35} - P_{13}P_{25} + P_{15}P_{23} = 0,$$
$$P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0.$$

The S_3 determined in S_9 by l_1, l_2, l_3, and a general point $l \equiv (l_{12} \cdots l_{45})$ of V_6^5 can be written parametrically:

$$y = \lambda(l_1) + \mu(l_2) + \delta(l_3) + \rho(l).$$

This meets V_6^5 in the points l_1, l_2, l_3, l, and a fifth point l', given by

* C. Segre, Enclykopädie der Mathematischen Wissenschaften, Band III, 2, Heft 7, p. 952.

$$
\text{(A)} \qquad \begin{aligned}
\lambda &= -\, l_{24}l_{25}(l_{12}l_{45} - l_{25}l_{34}),\\
\mu &= \quad l_{24}l_{45}(l_{12}l_{45} - l_{25}l_{34}),\\
\delta &= \quad l_{25}l_{45}(l_{12}l_{45} - l_{25}l_{34}),\\
\rho &= \quad l_{24}l_{25}l_{45}.
\end{aligned}
$$

Thus the equations of the transformation are:

$$
\text{(B)} \qquad \begin{aligned}
kl_{12}^1 &= l_{24}l_{25}^2 l_{34}, & kl_{24}^1 &= l_{24}^2 l_{25}l_{45},\\
kl_{13}^1 &= l_{13}l_{24}l_{25}l_{45}, & kl_{25}^1 &= l_{24}l_{25}^2 l_{45},\\
kl_{14}^1 &= l_{14}l_{24}l_{25}l_{45}, & kl_{34}^1 &= l_{12}l_{24}l_{45}^2,\\
kl_{25}^1 &= l_{25}^2 l_{45}(l_{14} - l_{34}), & kl_{35}^1 &= l_{45}l_{45}(l_{12} + l_{23}),\\
kl_{23}^1 &= l_{23}l_{24}l_{25}l_{45}, & kl_{45}^1 &= l_{24}l_{25}l_{45}^2.
\end{aligned}
$$

From equations (A) it is evident that the complex of invariant lines is given by

$$l_{12}l_{45} - l_{25}l_{34} = 0.$$

From (A) it also follows that the image of L_1 is any line of the special complex $l_{45}=0$; the image of L_2 is any line of the special complex $l_{25}=0$; and the image of L_3 is any line of the special complex $l_{24}=0$.

The singular lines are lines of the following systems:

1. Lines for which $l_{24} = l_{25} = l_{45} = 0$, (lines meeting T).
2. Lines for which $l_{34} = l_{35} = l_{45} = 0$, (lines meeting L_1).

2.1. Lines for which $l_{12} = l_{15} = l_{25} = 0$, (lines meeting L_2).

2.2. Lines for which $l_{12} + l_{23} = l_{24} = l_{14} - l_{34} = 0$, (lines meeting L_3).

3. Lines for which $l_{15} = l_{25} = l_{35} = l_{45} = 0$, (lines lying in the S_3 determined by L_1 and L_2).

3.1. Lines for which $l_{14} = l_{24} = l_{34} = l_{45} = 0$, (lines lying in the S_3 determined by L_1 and L_3).

3.2. Lines for which $l_{12} = l_{23} = l_{24} = l_{25} = 0$, (lines lying in the S_3 determined by L_2 and L_3).

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