ON SOME GAP THEOREMS FOR EULER'S METHOD OF SUMMATION OF SERIES

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Hardy and Littlewood* have proved the following theorem:

For a given series $\sum_{k=1}^{\infty} a_{n_k}$, $(a_{n_k} \neq 0)$, *let* θ *be a fixed constant such that*

$$\frac{n_{k+1}}{n_k} \geqq \theta > 1, \qquad (k = 1, 2, \cdots).$$

If this series be summable by Abel's method of summation to the sum s, *then this series is convergent and its sum is* s.

Obreschkoff† obtained also a similar result for Cesàro's method. We shall now study these results for Euler's method.

We shall begin with the following theorem:

THEOREM 1.‡ *Let* $\sum_{n=0}^{\infty} a_n$ *be a given series summable by Euler's method, that is, if* $s_0 = 0$, $s_n = a_0 + a_1 + \cdots + a_{n-1}$, $(n \geqq 1)$,

$$\lim_{n\to\infty} \frac{1}{2^n}\left\{ s_0 + ns_1 + \frac{n(n-1)}{2!} s_2 + \cdots + s_n \right\} = s \tag{1}$$

exists; and for two given increasing sequences $\{n_k\}$, $\{n_k'\}$, $(n_k < n_k')$, *of integers and for a given number* α, $(1 \leqq \alpha < 2)$, *let*

$$\begin{aligned} a_\nu &= 0, \text{ for } n_k < \nu < n_k', \quad (k = 1, 2, \cdots), \\ a_n &= O(\alpha^n). \end{aligned} \tag{2}$$

If $\eta_k'/\eta_k \geqq (1+\eta)/(1-\eta)$, $(k = 1, 2, \cdots)$, *for a positive number* η *such that*

$$(1+\eta)\log(1+\eta) + (1-\eta)\log(1-\eta) - 2\log\alpha > 0,$$

then

$$\lim_{k\to\infty} \sum_{\nu=0}^{n_k} a_\nu = s. \tag{3}$$

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* Hardy and Littlewood, Proceedings of the London Mathematical Society, (2), vol. 25 (1926).

† Obreschkoff, Tôhoku Mathematical Journal, vol. 32 (1930).

‡ If, in this theorem, (1) holds uniformly and O of (2) is independent of z when each a_n is a function of z, then (3) also holds uniformly.

PROOF. To prove this, we can consider that all $n_k' - n_k - 1$ are even. Then putting

$$n_k + \frac{n_k' - n_k - 1}{2} + 1 = m$$

we have

$$a_{m-1} = a_{m-2} = \cdots = a_{n_k+1} = 0,$$

$$a_m = a_{m+1} = \cdots = a_{n_k'-1} = 0.$$

Hence, if we put

$$s_n' = \frac{1}{2^n}\left\{s_0 + ns_1 + \frac{n(n-1)}{2!}s_2 + \cdots + s_n\right\},$$

then we have

$$\begin{aligned} s_{2m}' - s_m &= \frac{1}{2^{2m}}\left\{s_0 + 2ms_1 + \frac{2m(2m-1)}{2!}s_2 + \cdots + s_{2m}\right\} \\ &\quad - \frac{1}{2^{2m}}\left\{s_m + 2ms_m + \frac{2m(2m-1)}{2!}s_m + \cdots + s_m\right\} \\ &= \frac{1}{2^{2m}}\left\{-(a_0 + \cdots + a_{n_k}) - 2m(a_1 + \cdots + a_{n_k})\right. \\ &\quad - \frac{2m(2m-1)}{2!}(a_2 + \cdots + a_{n_k}) - \cdots \\ &\quad - \frac{2m(2m-1)\cdots(2m-n_k+1)}{n_k!}a_{n_k} \\ &\quad + \frac{2m(2m-1)\cdots(2m-n_k')}{(n_k'+1)!}a_{n_k} + \cdots \\ &\quad \left. + (a_{n_k} + \cdots + a_{2m-1})\right\}. \end{aligned}$$

Since from (2) we can find a positive constant M such that $|a_n| < Me^{n\log\alpha}$, $(n=0, 1, 2, \cdots)$, we get

$$\begin{aligned} |s_{2m}' - s_m| &< 2M\frac{e^{2m\log\alpha}}{2^{2m}}(n_k+1)^2\frac{2m(2m-1)\cdots(2m-n_k+1)}{n_k!} \\ &< 4M\frac{e^{2m\log\alpha}}{2^{2m}}\frac{2m\Gamma(2m)}{\Gamma(m+\lambda)\Gamma(m-\lambda)}, \end{aligned}$$

where $\lambda = m - n_k$.

Let us now put

$$f(m) = \frac{e^{2m\log\alpha}}{2^{2m}} \frac{2m\Gamma(2m)}{\Gamma(m-\lambda)\Gamma(m+\lambda)},$$

or

$$f(m) = \frac{e^{2m\log\alpha}}{2^{2m}} \frac{2m\Gamma(2m)}{\Gamma(m(1-\delta))\Gamma(m(1+\delta))},$$

where $\lambda = m\delta$, $(0<\delta=(n_k'-n_k+1)/(n_k'+n_k+1)<1)$. Then

$$\begin{aligned}\log f(m) &= 2m\log\alpha - 2m\log 2 + \log(2m) \\ &\quad + (2m-\tfrac{1}{2})\log(2m) - 2m + O(1) \\ &\quad - \{m(1-\delta)-\tfrac{1}{2}\}\log((1-\delta)m) + (1-\delta)m + O(1) \\ &\quad - \{m(1+\delta)-\tfrac{1}{2}\}\log((1+\delta)m) + (1+\delta)m + O(1) \\ &= -m\phi(\delta) + \tfrac{1}{2}\log m + O(1),\end{aligned}$$

where

$$\phi(\delta) = (1+\delta)\log(1+\delta) + (1-\delta)\log(1-\delta) - 2\log\alpha.$$

For a fixed number η_0, $(1>\eta_0\geqq 0)$, such that $\phi(\eta_0)=0$, any number η such that $1>\eta>\eta_0$ gives $\phi(\eta)>0$. When η is so fixed, it follows from (1) that

$$\lim s_m = \lim s_{2m}' = s \quad \text{for} \quad 1>\delta>\eta.$$

On the other hand, from $n_k'/n_k \geqq (1+\eta)/(1-\eta)$ we have

$$\frac{n_k' - n_k + 1}{n_k' + n_k + 1} > \eta.$$

Consequently $1>\delta>\eta$ since

$$\delta = (n_k' - n_k + 1)/(n_k' + n_k + 1).$$

Therefore

$$\lim_{k\to\infty} s_{n_k} = \lim_{m\to\infty} s_m = s.$$

Thus our theorem is completely proved.

REMARK. From Theorem I and Knopp's* theorem follows immediately Ostrowski's theorem:†

Let $f(z)=\sum_{n=0}^{\infty}a_n z^n$ *be a power series whose radius of convergence is* 1. *If*

$$a_\nu = 0 \quad for \quad n_k < \nu < n_k',$$

and

$$\frac{n_k'}{n_k} > 1 + \theta, \qquad (k = 1, 2, \cdots),$$

θ being a positive constant, then the partial sums s_{nk} of this series converge uniformly in a full neighbourhood of every regular point of the function of (z) on the unit circle.

THEOREM 2. *Let $\sum a_n$ be a given series summable by Euler's method to the sum s, and for two given increasing sequences $\{n_k\}$, $\{n_k'\}$, $(n_k<n_k')$, of positive integers and for $p \geqq -1$ let*

$$a_\nu = 0 \quad for \quad n_k < \nu < n_k', \quad (k = 1, 2, \cdots),$$
$$a_n = O(n^p).$$

If

$$\frac{n_k'}{n_k} \geqq 1 + c\left(\frac{3}{2} + p\right)^{1/2}\left(\frac{\log n_k}{n_k}\right)^{1/2}$$

holds for any given number c greater than 2, and for all sufficiently great integers k, then we have

$$\lim_{k\to\infty} \sum_{\nu=0}^{n_k} a_\nu = s.$$

PROOF. From the assumption we can take c' and q such that

$$\frac{n_k' - n_k}{n_k} \geqq c'\left(\frac{3}{2} + p + q\right)^{1/2}\left(\frac{\log n_k}{n_k}\right)^{1/2}, \qquad (c' > 2, q > 0).$$

Therefore

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* Knopp, Mathematische Zeitschrift, vol. 15 (1922).

† Zygmund (Journal of the London Mathematical Society, vol. 6 (1931)) proved Ostrowski's theorem similarly for the Borel method of summation.

$$\frac{n_k' - n_k}{n_k' + n_k} \geqq c'\left(\frac{3}{2} + p + q\right)^{1/2}\left(\frac{\log n_k}{n_k}\right)^{1/2}$$

$$:\left(c'\left(\frac{3}{2} + p + q\right)^{1/2}\left(\frac{\log n_k}{n_k}\right)^{1/2} + 2\right)$$

$$\geqq \left(\frac{3}{2} + p + q\right)^{1/2}\left(\frac{\log n_k}{n_k}\right)^{1/2}$$

for all sufficiently great integers k. Hence from

$$\delta = \frac{n_k' - n_k + 1}{n_k' + n_k + 1} > \frac{n_k' - n_k}{n_k' + n_k}$$

we get

$$\delta^2 \geqq \left(\frac{3}{2} + p + q\right)\frac{\log m}{m}, \qquad \left(m = \frac{n_k' + n_k + 1}{2}\right).$$

Consequently, from

$$(1 + \delta)\log(1 + \delta) + (1 - \delta)\log(1 - \delta) > \delta^2,$$

we have

$$(1 + \delta)\log(1 + \delta) + (1 - \delta)\log(1 - \delta) > \left(\frac{3}{2} + p + q\right)\frac{\log m}{m},$$

whence as in the proof of Theorem I we obtain

$$\begin{aligned} |s_{2m}' - s_m| &< Mm\frac{(2m)^p}{2^{2m}}\frac{\Gamma(2m)}{\Gamma(m(1-\delta))\Gamma(m(1+\delta))} \\ &< M' \exp\left[m\left\{\left(\frac{3}{2} + p\right)\frac{\log m}{m} - (1+\delta)\log(1+\delta)\right.\right. \\ &\qquad \left.\left. - (1-\delta)\log(1-\delta)\right\} + O(1)\right] \\ &\to 0, \quad (m \to \infty), \end{aligned}$$

M, M' being constants. Therefore

$$\lim s_{n_k} = \lim s_m = s.$$

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