A NOTE ON MATRICES DEFINING TOTAL REAL FIELDS*

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Let K be algebraic of degree n over a sub-field F of the field of all real numbers. Then there is an equation

$$f(x) = x^n + a_1x^{n-1} + \cdots + a_n = 0, \qquad (a_i \text{ in } F), \tag{1}$$

which is irreducible in F, and $K = F(X)$ consists of all polynomials with coefficients in F in an algebraic quantity X for which $f(x) = 0$. We call K a *total real* field over F if the ordinary complex roots

$$x_1, \cdots, x_n \tag{2}$$

of $f(x) = 0$ are all real. The modern theory of algebraic numbers has made the study of such fields of great interest.

A particular algebraic root of $f(x) = 0$ is given by the matrix

$$Y = \begin{pmatrix} 0 & 0 \cdots 0 & -a_n \\ 1 & 0 \cdots 0 & -a_{n-1} \\ 0 & 1 \cdots 0 & -a_{n-2} \\ \cdot & \cdot \cdot \cdot \cdot \cdot & \cdot \\ 0 & 0 \cdots 1 & -a_1 \end{pmatrix}. \tag{3}$$

This is a matrix whose characteristic equation is the above $f(x) = 0$. The irreducibility of $f(x)$ implies that every n-rowed square matrix Z with elements in F and $f(x) = 0$ as characteristic equation is similar to Y, and thus every such Z defines a field $F(Z)$ equivalent to K over F.

We shall obtain a normal form here for Z such that every Z in our form and with irreducible characteristic equation defines a total real field, while conversely every total real field is defined by one of our matrices. Our result will then provide *a construction of all total real fields over F.* The irreducibility condition is of course a part of the final conditions in all problems on the construction of algebraic fields and should not be considered as

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affecting the completeness of our criterion. We shall in fact prove the following theorem.

THEOREM. *Let D be an n-rowed diagonal matrix with positive diagonal elements in F, and S be any symmetric n-rowed square matrix with elements in F for which the characteristic function of*

$$Z = DS \tag{4}$$

is irreducible in F. Then $F(Z)$ is a total real field of degree n over F. Conversely every total real field K of degree n over F is equivalent to a field $F(Z)$ with Z given by (4).

For if D and E are the n-rowed diagonal matrices*

$$D = \operatorname{diag}\{d_1, \cdots, d_n\}, \quad e_i = d_i^{1/2}, \quad E = \operatorname{diag}\{e_1, \cdots, e_n\}, \tag{5}$$

then $D=E^2$, $E=E'$ is a real symmetric matrix,

$$E^{-1}ZE = E^{-1}E^2SE = ESE'$$

is a real symmetric matrix. Thus the characteristic roots of $E^{-1}ZE$ are all real. But they are the roots of the characteristic equation of Z and we are assuming that this equation is irreducible in F. Hence $F(Z)$ is a total real field.

Conversely let K be total real of degree n over F so that K is equivalent over F to $F(Y)$ with Y given by (3). We let V be the Vandermonde matrix

$$\begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & x_n & \cdots & x_n^{n-1} \end{pmatrix}. \tag{6}$$

The square of the determinant of V is the discriminant of $f(x)$ and is not zero when $f(x)$ is irreducible in F. This is our hypothesis, so that the matrix

$$T = V'V = (s_{i+j-2}), \qquad (i, j = 1, \cdots, n), \tag{7}$$

is non-singular. Also the symmetric function $s_k = \sum_{g=1}^{n} x_g^k$ is well known to be a polynomial in $a_1, \cdots, a_n$ with integral coeffi-

* We use the notation diag $\{d_1, \cdots, d_n\}$ to mean an n-rowed square matrix whose elements off the principal diagonal are zero and whose principal diagonal is $d_1, \cdots, d_n$.

cients, so that T has elements in F. Since V is a *real* non-singular matrix, the matrix $T=V'V$ is *positive definite* symmetric. This is actually the true reason for our result.*

There exists a non-singular B with elements in F such that

$$B'TB = \begin{bmatrix} g_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & g_n \end{bmatrix}, \qquad (g_i \text{ in } F). \tag{8}$$

Since T is positive definite, so is $B'TB$, and the g_i must be positive. Thus

$$D = \text{diag}\,\{d_1, \cdots, d_n\}, \qquad d_i = g_i^{-1} > 0,\ D^{-1} = B'TB. \tag{9}$$

By an elementary computation

$$VYV^{-1} = \text{diag}\,\{x_1, \cdots, x_n\}. \tag{10}$$

The diagonal matrix VYV^{-1} is symmetric and

$$(VYV^{-1})' = (V')^{-1}Y'V' = VYV^{-1}, \quad (V'V)Y = Y'(V'V). \tag{11}$$

Hence $TY=Y'T$, $(B'TB)B^{-1}YB=B'Y'(B')^{-1}B'T'B$, whence

$$D^{-1}Z = Z'(D^{-1})', \qquad Z = B^{-1}YB. \tag{12}$$

The matrix $S=D^{-1}Z$ is now symmetric since $S'=Z'(D^{-1})'=S$. Then

$$Z = DS$$

as desired, and our theorem is proved.

Notice in closing that the positive elements of the matrix D are the inverses of the elements in the diagonal normal form of the *discriminant matrix* T. When this normal form of T is the identity matrix the result Z is a symmetric matrix S for which the total reality of $F(Z)$ is a classical result.†

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* See Bieberbach-Bauer, *Vorlesungen über Algebra*, 1933, p. 184, for the known theorem stating that T is positive definite when $f(x)=0$ has all real roots. That this result is true is an evident consequence of the definition of positive definiteness.

† See L. E. Dickson, *Modern Algebraic Theories*, 1926, p. 76; Theorem 12.