ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

94. Professor G. C. Evans: *Equilibrium problems for potentials of positive and negative mass.*

The problem of equilibrium potentials for positive and negative masses and a power of the distance less than one involves considerations which do not enter in typical calculus of variations investigations. That the general problem is not illusory is evidenced by the existence of a minimum energy distribution in the case of concentric spherical shells of positive and negative mass, respectively, whose total amounts differ by unity. Both masses and radii are variable. (Received January 12, 1937.)

95. Mr. A. N. Milgram: *The generalized Mullikin theorem.*

It is known that, in the plane, the sum of a countable number of closed sets, no one of which separates the plane and whose mutual intersections are vacuous, has a connected complement. This is however seen to be a particular case of the following theorem which is demonstrated in this paper: The sum of a countable number of closed sets, no one of which separates R^n (that is, euclidean n-space) and for which the intersection of any pair is of dimension at most $n-3$, has a connected complement in R^n. The proof is based upon two other theorems. If F is a closed set of dimension at most $n-r-2$, and C^{r+1} an algebraic complex in R^n such that $\dot{C}^{r+1}$ is contained in R^n-F, then for $\epsilon>0$ there exists a complex C_1^{r+1} contained in $S\ (\epsilon, C^{r+1})\times(R^n-F)$ for which $\dot{C}^{r+1}=\dot{C}_1^{r+1}$. It is next demonstrated that if Γ^1 is the simple closed curve $axbya$, and $\dot{C}^2=\Gamma^1$, while F is a closed set for which $C^2-F=M_1+M_2$ where M_1 and M_2 are mutually separated and $M_1\supset a$, $M_2\supset b$, then F contains a connected set having a non-vacuous intersection with the arcs axb and ayb. (Received January 12, 1937.)

96. Professor R. P. Agnew: *Comparisons of products of methods of summability.*

In terms of two methods A and B of summability with matrices (a_{nk}) and (b_{nk}), two types of products are defined. A sequence $\{s_n\}$ is summable to L by the *iteration product* AB if $V_n\to L$ where $U_n=\sum_{p=1}^{\infty}\sum_{k=1}^{\infty}a_{np}b_{pk}s_k$; and is summable to L by the *composition product* $A\cdot B$ if $V_n\to L$ where $V_n=\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}a_{np}b_{pk}s_k$. Relative inclusion, equivalence, and consistency of pairs selected from the four methods A, B, AB, and $A\cdot B$ are discussed. The desirability of having different names and notations for the two products is evidenced by the fact, proved in this paper, that AB and $A\cdot B$ may be inconsistent methods of summability even when A and B are regular and $a_{nk}\geqq 0$, $b_{nk}\geqq 0$. (Received January 16, 1937.)

97. Dr. R. P. Bailey: *A note on the convergence of linear operations.*

Call a set of points in a normed vector space a *p-set* provided it is true that if x and y are any elements of this set, $\|x-y\|\leqq\|x+y\|$. Let S denote a sequence $\{U_n(x)\}$ $(n=1, 2, \cdots)$ of linear operations defined over the elements $\{x\}$ of a Banach space B_1, with values in a second Banach space B_2. The following theorem is established: *For the convergence of S over B_1, it is necessary and sufficient that S converge over a set dense in a sphere k of B_1 such that for n sufficiently large the transforms $\{U_n(x)\}$ $(x\subset k)$ constitute a p-set in B_2.* The effect of the theorem is to exhibit in a new and clearer light the convergence properties of certain special sequences of operations, in particular the so-called "positive" functionals. (Received January 20, 1937.)

98. Mr. Garrett Birkhoff: *On lattices of hyperplanes.* I.

The subspaces of any vector space over a (not necessarily commutative) field F form a modular lattice—more exactly, a projective geometry. Sublattices of this lattice which contain the origin O and the whole space I are therefore also modular lattices—called lattices of hyperplanes. It is shown that in any modular lattice L of finite dimensions, the possible "dimension functions" $d[x]$ making $d[x\cup y]\geqq d[x]$ and $d[x]+d[y]=d[x\cup y]+d[x\cap y]$, are the linear combinations $d[x]=d[0]+\sum_p\mu_p d_p[x]$ (with $\mu_p\geqq 0$) of the "prime" dimension functions $d_p[x]$ expressing the number of occurrences of each prime quotient p in chains from 0 to x. The congruence relations θ on L are had by making a finite subset of the μ_p be zero, and $x\sim y$ (θ) means $d[x\cup y]=d[x\cap y]$. Thus they are a Boolean algebra. An element x is "neutral" if and only if, for all p, either $d_p[x]=d_p[0]$ or $d_p[x]=d_p[I]$. Therefore any complement of a neutral element is neutral, and unique if it exists at all. There exist modular lattices which are not "primary" (have more than one p), but have no non-trivial neutral elements. (Received January 15, 1937.)

99. Dr. D. M. Dribin (National Research Fellow): *A note on groups of order* 24.

Let V represent the "Vierergruppe," and let S_3 and S_4 represent the symmetric groups of degree 3 and 4, respectively. In the present brief note, it is shown that there are but three abstract groups G of order 24 having V and S_3 as subgroups, with V invariant in G, and with G/V isomorphic to S_3. These groups are (a) S_4, (b) the direct product of V and S_3, and (c) a group defined abstractly in terms of its generators A, B, Q, as follows: $A^4=B^2=Q^3=1$, $BAB=A^{-1}$, $A^{-1}QA=Q^{-1}$, $BQ=QB$. The results are of importance in the theory of quartic fields with the symmetric group. (Received January 16, 1937.)

100. Mr. E. H. Larguier: *The schools of thought in modern mathematics.*

After outlining briefly the main points of several schools of thought in the modern foundations of mathematics, the author offers a critical study from the view point of scholastic philosophy of certain philosophical implications of the concepts used, in particular *starting-point* and *truth.* This paper will appear shortly in Thought (America Press, N. Y.). (Received January 16, 1937.)

101. Dr. D. H. Lehmer: *On the Hardy-Ramanujan series for the partition function.*

Hardy and Ramanujan have given an asymptotic series for the number of unrestricted partitions of n which gives the values of this numerical function with extraordinary accuracy. Because of the complicated nature of the coefficients of the series they could not decide whether the series is convergent or not. In this paper the author proves that the series is actually divergent for every n. This result follows from a study of the properties of the coefficients and is accomplished by exhibiting an infinite sequence of terms of the series which do not tend to zero. It is shown incidentally that for some values of n the coefficients are unbounded. (Received January 22, 1937.)

102. Professor Deane Montgomery: *Almost periodic transformation groups.*

In view of the recent work on topological groups it is natural to consider the situation which arises when such groups act as transformation groups on various types of spaces. A study of this situation is begun here from the point of view of almost periodic transformation groups the definition of which is suggested by von Neumann's definition of almost periodic functions in a group (Transactions of this Society, vol. 36, p. 445). For a wide class of spaces compact transformation groups are a special case of almost periodic transformation groups. This paper concerns itself chiefly with the nature of the minimal closed invariant sets of the group. One of the principal theorems states that if a compact one dimensional group acts on three-space in such a way that all orbits are uniformly bounded in diameter then every point of the space is fixed under the group. Hence if such a group is to act in a non-trivial manner the diameters of its orbits must be unbounded. (Received January 22, 1937.)

103. Dr. A. E. Pitcher: *Critical points of a pair of functions.* Preliminary report.

Marston Morse has developed a very comprehensive theory of critical points of a function. This paper is the beginning of an extension of this theory to pairs of functions. Let f and g be two functions of class C^2 in the variables $(x)=(x_1, \cdots, x_n)$ in a region. The point (x^0) is termed a *critical point* of f, g if the two-rowed matrix f_i, g_i, is not of maximum rank at (x^0). This definition is independent of changes of coordinates of class C^2. Two sets of necessary and sufficient conditions that (x^0) be a critical point of f, g are obtained in terms of multipliers. When f and g are defined on an analytic Riemannian manifold of dimension 2 and are analytic in terms of admissible local coordinates, the critical points of f, g form a 1-dimensional complex whose 1-cells are simple analytic arcs. The author discusses the topology of this complex and some of its subsets. Wide use is made of the methods developed by Morse. Part of the results (for $n=2$, but not for $n>2$) probably could be obtained from a recent work of A. W. Tucker (this Bulletin, vol. 42 (1936), 859–862). (Received January 23, 1937.)

104. Dr. M. S. Robertson: *Multivalent functions of order p.*

Let $f(z)=\sum_{n=0}^{\infty} a_{kn+1}z^{kn+1}$ be holomorphic and multivalent of order p for $|z|<1$. Thus $f(z)$ takes on no value more than p times and does take on some value p times within the unit circle. If k is a positive integer less than $4p$ the author shows that $a_n=O(n^{(2p/k)-1})$. (Received January 22, 1937.)

105. Dr. I. J. Schoenberg: *On certain metric spaces arising from euclidean spaces by a change of metric and their imbedding in Hilbert space.*

From the m-dimensional euclidean space R_m a new metric space $\mathcal{R}_m^{(\gamma)}$ is obtained by passing from the euclidean distance $\overline{PP'}$ to the new distance $d(P, P')=\overline{PP'}^{\gamma}$, $(0<\gamma<1)$. It is proved by analytical methods that $\mathcal{R}_m^{(\gamma)}$ can be imbedded isometrically in the Hilbert space. A special case of this theorem, for $m=1$ and $\gamma=1/2$, was previously established by W. A. Wilson, American Journal of Mathematics, vol. 57 (1935), p. 64. (Received January 20, 1937.)

106. Dr. Abraham Sinkov: *On generating the simple group $LF(2, 2^N)$ by two operators of periods two and three.* Preliminary report.

By considering the possible automorphisms of $LF(2, 2^N)$, the number of abstractly distinct ways in which this group may be generated by two operators of periods two and three is found to be equal to at most $(1/N)\{2^N-\sum 2^{N/p_1}+\sum 2^{N/(p_1p_2)}-\cdots(-1)^s 2^{N/(p_1p_2\cdots p_s)}\}$ where the p_i are the distinct prime divisors of N. It is next shown that if the periods of the product and commutator of any pair of generators are n and p respectively, then these generators satisfy the relations $G^{3,n,p}$. As a result, to every pair of generators satisfying $(2, 3, n; p)$ $[n\neq p]$, there corresponds a second pair satisfying $(2, 3, p; n)$. Hence the totality of values which is assumed by n in the various definitions of any particular $LF(2, 2^N)$ is identical with the totality of values assumed by p. (Received January 27, 1937.)

107. Mr. W. C. Strodt: *Systems of algebraic partial difference equations.*

This paper, using a process developed by J. F. Ritt for algebraic differential equations, and modified by Ritt and J. L. Doob for ordinary algebraic difference equations, proves the following theorem: If Σ is any system of algebraic difference equations in n unknowns and m variables, then the manifold of solutions of Σ is the sum of the manifolds of a finite number of irreducible systems. This decomposition is essentially unique. (Received January 14, 1937.)

108. Mr. A. L. Whiteman: *On a generalized theorem of higher reciprocity.*

Let $\mathfrak{D}$ denote the totality of polynomials in an indeterminate, x, with coefficients in a Galois field $GF(p^n)$ of order p^n. Let P, Q be primary irreducible polynomials in $\mathfrak{D}$ of degrees ν and ρ respectively. Then if A is any polynomial in $\mathfrak{D}$ not divisible by P, define $\{A|P\}$ as that element of $GF(p^n)$ for which $\{A|P\} \equiv A^{(p^{n\nu}-1)/(p^n-1)}$ (mod P). One has then the following theorem of reciprocity due to H. Kuhne (rediscovered by Schmidt and Carlitz): $\{P|Q\} = (-1)^{\rho\nu}\{Q|P\}$. In this paper Kuhne's theorem is extended to the case when P and Q are relatively prime polynomials. A short proof is given which depends upon an extension of the analog of Gauss's lemma. (Received January 16, 1937.)

109. Mr. Garrett Birkhoff: *On lattices of hyperplanes.* II.

Let $PG(F; n-1)$ be the projective geometry of all right-subspaces of the linear space of n-vectors over a (not necessarily commutative) field F. To obtain the most general equidimensional complemented sublattice CSL of $PG(F; n-1)$, (1) take any partition $n = k_1 + \cdots + k_h$ of n, (2) assign to each $k_i = 2$ a subset of F containing more than one element, (3) assign to each $k_i > 2$ a subfield S_i of F, (4) transform coordinates. CSL is a projective geometry (or *subgeometry*) if and only if $h = 1$; otherwise it is reducible. In the first case, if $n > 2$, CSL is isomorphic with $PG(S_1; n-1)$; in the general case, CSL is the direct sum of irreducible constituents, each of which is a $PG(S_i; k_i-1)$ if $k_i > 2$. Again, if $n = km$, and F^* is any extension of F of degree k, then $PG(F; n-1)$ contains $PG(F^*; m-1)$ as a subgeometry. Since the commutativity of F does not imply that of F^*, a Pascalian geometry may thus contain a non-Pascalian subgeometry. On the other hand, every subgeometry of a $PG(F; n-1)$ is Desarguesian. (Received January 22, 1937.)

110. Mr. Garrett Birkhoff: *The insolubility of quintic equations.*

A new proof is given of the theorem that there is no general solution in terms of radicals to the equation of fifth degree. The proof rests upon consideration of the monodromie groups of algebraic functions of n complex variables. It is shown that the monodromie groups of all functions involving only radicals and rational operations are solvable, whereas the monodromie group of the general solution of $z^5 + a_1z^4 + \cdots + a_5 = 0$ is the symmetric group of degree five, and hence insolvable. (Received March 2, 1937.)

111. Professor A. B. Brown: *Critical curvatures in Riemannian spaces.*

The normal curvatures of curves through a point on a surface in 3-space are known to be either constant, independent of the direction, or else have two critical values taken on in perpendicular directions, the smaller value being a minimum and the larger a maximum. This result is generalized directly to the case of an n-surface in a Riemannian $(n+1)$-space with positive definite fundamental form, and also to an n-surface in a Riemannian space of any higher dimension, where in this case the curvature vectors are projected on any direction normal to the surface. Finally, a similar result applies to the Ricci mean curvatures at a point of a Riemannian n-space. The generalization is as follows: the number of critical values is at most, and in general exactly, n. When it is n, the values are taken on only in n (mutually perpendicular, as is well known) directions, and if arranged according to algebraic values, the ith is at a non-degenerate critical point of index $i-1$. For use in the proofs, Riemannian coordinates are established for the case that the coefficients of the fundamental form are of class C^4, in place of the usual assumption of analyticity. (Received February 24, 1937.)

112. Professor A. B. Brown: *Critical diameters of central quadrics.*

If a central quadric surface in n-space, $n>1$, has only a finite number of critical directions for the lengths of its diameters (in particular, if it is not a surface or of revolution), the ith in length is at a non-degenerate critical point of index $i-1$. Central quadrics are so defined that if $n=2$ they are: circle, ellipse, hyperbola, pair of parallel straight lines. (Received March 1, 1937.)

113. Professor S. S. Cairns: *On polyhedral manifolds and differentiable manifolds.*

Let P_r denote a complex made up of simplexes in a euclidean n-space, E_n. Suppose the star of every vertex of P_r is an r-cell and can be mapped by a piecewise linear homeomorphism (that is, linear on each r-simplex of the star) onto a set of simplexes in an auxiliary r-space. It is then proved that, if n is sufficiently large, it is possible, by arbitrarily small displacements of the vertices, to bring P_r into a position such that the star of any k-simplex, s_k, of P_r has a one-to-one orthogonal projection onto some r-plane through s_k. Among other possible applications of this theorem, the author notes that it prepares the way for a method of constructing a differentiable approximation to P_r by a process of rounding off the edges and corners. The details of this construction are not completed in the most general case. (Received March 5, 1937.)

114. Mr. J. W. Calkin: *General self-adjoint boundary conditions for second order partial differential operators of elliptic type.* Preliminary report.

Let E be the set of points in the (x, y)-plane interior to a regular closed curve C. Define, in the complex Hilbert space $\mathfrak{L}_2(E)$, a transformation T: $Tf=-(pf_x)_x-(pf_y)_y+qf$; p_{xx}, p_{yy}, q bounded measurable, $p>0$, $\mathfrak{J}(q)=0$, on $E+C$. The domain $\mathfrak{D}^*$ of T is defined by explicit local conditions on its elements and the implicit condition $\int_E |Tf|^2dE<\infty$. H is defined equal to T on $\mathfrak{D}$, the set of elements g of $\mathfrak{D}^*$ which satisfy the conditions $g(s)=\partial g/\partial n=0$ almost everywhere on C. Let W be a closed isometric transformation in $\mathfrak{L}_2(C)$ such that, for a dense set of elements h in the domain of W, the equations $f(s)=h-Wh$, $-p\partial f/\partial n=i(h+Wh)$ have a solution f in $\mathfrak{D}^*$. Let $H(W)$ be the transformation in H^*, whose domain is the set of all such solutions. Then

$T^* \equiv H \subset H(W) \subset \tilde{H}(W) \subset H^*(W) \subset H^* \equiv T$. The author determines conditions on W, necessary and sufficient for $H(W)$ to belong to a specified "large" subclass of the class of all self-adjoint extensions of H. (That W be unitary is neither necessary nor sufficient for $\tilde{H}(W) \equiv H^*(W)$ to be true.) Also, if $W \equiv I$, or if $W - I$ has a bounded inverse, then $H(W)$ *is defined and is itself self-adjoint*, bounded below, and has a resolvent with kernel of Hilbert-Schmidt type. In establishing these results, the fact is used that the Dirichlet and Neumann problems of potential theory are solvable for E, but no other previous solutions of boundary value problems are employed. (Received March 4, 1937.)

115. Professor Leonard Carlitz: *Higher congruences and irreducible polynomials.*

In this note higher congruences of certain types are completely factored into irreducible components. Some special cases have occurred in the author's *Criteria for certain higher congruences* to appear in the American Journal of Mathematics. (Received March 5, 1937.)

116. Professor Leonard Carlitz: *The singular series for polynomials.*

In this paper certain additive problems for polynomials in a Galois field are discussed. The discussion depends upon a "singular series" analogous to the Hardy-Littlewood series; particular cases have been treated in an earlier paper (abstract 42-7-313). (Received March 5, 1937.)

117. Professor Richard Courant: *Soap film experiments on the problem of Plateau and Douglas.*

By some soap film experiments it can be demonstrated that in certain cases Plateau's problem for given contours may have more than one solution. In particular with flexible boundaries one can illustrate how certain solutions become unstable and change discontinuously into solutions of a quite different topological type. It also can be demonstrated that sometimes minimal surfaces are stable which do not give the least area. (Received March 9, 1937.)

118. Professor H. B. Curry: *On the paradox of Kleene and Rosser.*

Kleene and Rosser showed in 1935 (Annals of Mathematics, vol. 36, pp. 630–636; the proofs depend on a series of previous papers) that certain systems of formal logic are inconsistent in the sense that every formula expressible in their notation can be proved. Their argument was a modification of the Richard paradox. The object of the present new derivation is to bring out clearly that the paradox depends essentially on two properties of the system: (1) the fact that an arbitrary expression involving x can be exhibited as a function of x in the system, and (2) the presence of a deduction-theorem (analogous to the theorem so called in Hilbert and Bernays, *Grundlagen der Mathematik*, I, 1934, p. 151). In some respects, also, the new derivation is simpler. (Received March 5, 1937.)

119. Professor E. L. Dodd: *Regression coefficients as means of certain ratios.*

Suppose that each of the variables, $x, y, \cdots, u, w$, takes on n values: $x_1, x_2, \cdots, x_n; \cdots; w_1, w_2, \cdots, w_n$. Let $\omega = a + bx + \cdots + ku$ be an estimate of w, with coefficients determined by the least square method, that is, by minimizing $\sum(\omega_i - w_i)^2$. Then b is a mean of the ratios w_i/x_i; k is a mean of the ratios w_i/u_i. By a mean F of $t_1, t_2, \cdots, t_n$ is understood a function $F(t_1, t_2, \cdots, t_n)$ such that if c is any value that each t can take on, then $F(c, c, \cdots, c) = c$. The above b, as a mean of w_i/x_i, may be either *internal* or *external*. Generalization is made to non-linear regression equations. (Received February 19, 1937.)

120. Dr. Aaron Fialkow (National Research Fellow): *Isometric invariants of systems of curves on a surface.*

This paper considers a one-parameter system of curves, $f(u, v) = c$, on an orientable surface. The orthogonal trajectories of these curves are drawn, forming an orthogonal net. Then at each point of the surface there is defined the geodesic curvatures of the two curves passing through the point as well as their tangential and normal derivatives. It is assumed that $f(u, v)$ satisfies an isometrically invariant partial differential equation involving u and v and the components (and their derivatives) of the metric tensor of the surface. Then the family of curves will have geometric characterizations which may be stated as relations among the geodesic curvatures and directional derivatives of the curvatures of the associated orthogonal net. Such a characteristic property is obtained for the differential equation $\Delta_2 f = F(f)$ and an additional relation if $F(f)$ specializes to a linear function of f. Applications of these results are made to problems in differential geometry and to the heat equation on a surface and the wave equation for a vibrating membrane. The author also finds certain geometric relations which are identities. The simplest of these expresses the Gaussian curvature of the surface in terms of the geodesic curvatures of the net. (Received March 3, 1937.)

121. Dr. M. M. Flood: *Equivalence of pairs of matrices.*

Kronecker was the first to obtain necessary and sufficient conditions for the equivalence of two pairs of matrices. The purpose of this paper is to present an elementary and completely rational treatment of this problem. The method of proof is like that used by Ingraham in his construction of a canonical form for singular pairs of hermitian matrices. The invariants which are shown to characterize a pair of matrices are essentially the ones considered recently by Turnbull, and later by Williamson, who showed that Kronecker's invariants can be expressed in terms of these new invariants. (Received March 2, 1937.)

122. Mr. M. A. Girshick: *Sampling distributions related to determinantal equations.*

In this paper the joint distribution of canonical correlations (see Hotelling, *Relations between two sets of variates*, Biometrika, vol. 28 (December, 1936)), in the case of two independent sets of variates with s variates in the first set and t in the second, is reduced to a product of two functions; a known function involving the number in the sample, and an unknown function independent of it. From this the joint moments of q and z are obtained together with the actual distribution of the canonical correlations in the case $s=2$ and t general. The latter distribution is shown to approach in the limit a generalized chi-square form which is identical with the joint sampling distributions of the two latent roots of a determinant of sample covariances (with t items in each sample) of two normally and independently distributed variates with unit variance in the population. The author also finds the asymptotic variances and covariances of latent roots of a determinant of sample covariances of p normally distributed variates and hence deduces the limiting distributions of these roots. (Received March 4, 1937.)

123. Dr. J. A. Greenwood: *A matching problem.*

A given target deck consists of s distinct sets of r identical cards each. Another like deck of cards is shuffled and placed face down in one-to-one correspondence with the first deck. The probability of obtaining exactly i "hits" or "matchings" is obtained ($i=0, 1, \cdots, rs$) involving a reduction formula. If either r or s equals 2 the probability is given directly without recourse to a reduction formula. An interesting contribution is the application of determinant notation to the problem. A trivial proof is also given that the theoretical average number of "hits" per trial is r, thus establishing the identity $\sum_{i=0}^{rs} ip_i = r$, where p_i is the probability of obtaining exactly i hits in one trial. (Received March 4, 1937.)

124. Dr. Marshall Hall: *Group rings and extensions.* I.

The problem of extending an abelian group A by any finite group H is translated by a fundamental theorem into a problem on H, the group ring of H. Necessary and sufficient conditions on factors sets (elements in A) for determining extensions are shown to be paraphrases of identities on defining relations of H. These, in turn, are equivalent to certain linear equations in H. All extensions of A by H are determined, and the group of factor sets is characterized. As one application of these results, a new criterion is given for direct factors (not necessarily abelian), of which certain classical criteria are special cases. Further applications will be made in a later paper. (Received March 4, 1937.)

125. Dr. Israel Halperin: *Superimposition of decompositions in continuous geometries.*

Continuous geometries were recently discovered as an extension of projective geometry by J. von Neumann. The remarkably elegant theory developed by him gives an essentially unique dimensionality function if the geometry is irreducible (see J. von Neumann, Proceedings of the National Academy of Science, 1936, pp. 92–101). For geometries in which irreducibility is not assumed the following theorem is here shown: Let $b=\sum_{\alpha\epsilon J} a_\alpha=\sum_{\beta\epsilon K} c_\beta$ be two decompositions of the same element b. Then there exist direct refinements $b=\sum^{+}_{\gamma\epsilon I}\bar{a}_\gamma=\sum^{+}_{\gamma\epsilon I}\bar{c}_\gamma$, $\bar{a}_\gamma\leqq a_\alpha$, $\bar{c}_\gamma\leqq c_\beta$ for some $\alpha=\alpha(\gamma)$, $\beta=\beta(\gamma)$, such that $\bar{a}_\gamma$

and $\bar{c}_\gamma$ are perspective for every γ. This result is used to obtain an essentially unique dimensionality function for reducible geometries with suitable sets of partial-automorphisms. It is also shown that perspectivity is unrestrictedly additive. (Received March 5, 1937.)

126. Mr. B. A. Hausmann and Professor Oystein Ore: *Theory of quasi-groups.*

In this paper the structural properties of systems with one (non-associative) operation are considered. It is shown that the requirement that some of the principal decomposition properties of groups, like existence of co-set decompositions or normal subsets, shall hold leads to various formulations of weak associative laws. Systems with such laws are investigated. In certain cases they may be derived from ordinary groups by operations on the group elements. (Received March 4, 1937.)

127. Professor Einar Hille: *On the inversion problem of Möbius.*

This paper contains a study of a number of analytical problems connected with the algorithm of Möbius, that is, the bilinear system of equations $a_1b_1=1$, $\sum_{d|n}a_kb_{n|d}=0$, $n>1$. Among these problems should be mentioned Möbius' original inversion problem for power series, the reciprocation of ordinary Dirichlet series, and the solution of the functional equation $\sum_1^\infty a_nF(nz)=G(z)$. The problem of solving the equation $\int_0^\infty F(uz)dA(u)=G(z)$ leads to analogous algorithms connected with the problem of expressing the reciprocal of a Laplace integral as an integral of the same type. (Received March 4, 1937.)

128. Dr. Nathan Jacobson (National Research Fellow): *Simple Lie algebras of type A.*

In an earlier paper (abstract 43-1-74) the author studied the Lie algebras defined by J-skew elements of a normal simple self-reciprocal associative algebra. In the present paper the author considers a simple associative algebra $\mathfrak{A}$ of order $2n^2$ over Φ of characteristic 0 with centrum $\mathrm{P}=\Phi(c)$, $c^2=\gamma\epsilon\Phi$ and such that $\mathfrak{A}$ has an involutorial anti-automorphism J such that $(\xi a)^J=\bar{\xi}a^J$ for $\xi\epsilon\mathrm{P}$. If $n>2$ the set $\mathfrak{S}_J$ of J-skew elements forms a Lie algebra relative to commutator multiplication and the derived algebra $\mathfrak{S}_J'$ is normal simple of type A_{II} as defined by Landherr (Hamburg Abhandlungen, vol. 11, p. 50) and conversely any Lie algebra of type A_{II} can be obtained in this way. The algebras $\mathfrak{S}_{J_1}'$ and $\mathfrak{S}_{J_2}'$ defined by J_1 and J_2 in $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic if and only if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic and J_1 and J_2 are cogredient ($J_2=S^{-1}J_1S$, S an automorphism). The author determines also the group of automorphisms of any $\mathfrak{S}_J'$ and discusses the question of cogredience of involutorial anti-automorphism in relation to ordinary cogredience of hermitian matrices with elements in a self-reciprocal division algebra. (Received March 1, 1937.)

129. Dr. F. B. Jones: *On cyclic connectivity and local end points.*

In this paper the author has extended several of the known results about cyclic connectivity. The main result for spaces containing no cut point is the following theorem: in order that a space satisfying axioms 0–3 of R. L. Moore's *Foundations of Point Set Theory* be cyclicly connected, it is necessary and sufficient that it contain no local end points. (Received February 19, 1937.)

130. Dr. D. H. Lehmer: *On the coefficients of the series for the partition function.*

In 1917 Hardy and Ramanujan gave a remarkable asymptotic series for $p(n)$, the number of partitions of n, whose kth coefficient $A_k(n)$ was given as a complicated sum of $24k$th roots of unity associated with the elliptic modular function generating $p(n)$. Recently Rademacher has given a convergent series for $p(n)$ having these same coefficients. The apparent complexity of A_k not only prevented Hardy and Ramanujan from making a study of A_k for large k, thus leaving open the question of when the series could be used, but also made the calculation of more than a few terms of either series a difficult matter. This paper makes a study of these coefficients. Multiplication theorems are proved which exhibit A_k as a product of A's whose subscripts are the prime power divisors of k. If k is a prime power, $A_k \cdot k^{-1/2}$ is merely twice the real part of a definite root of unity. These results obviate the necessity of tables of A_k and also permit the accurate estimation of A_k for large and small k. Thus it is possible to give comparatively accurate expressions for the remainders of the Hardy-Ramanujan and Rademacher series. (Received March 4, 1937.)

131. Dr. Saunders MacLane and Professor V. W. Adkisson: *Fixed points and the extension of the homeomorphisms of a planar graph.*

The authors propose to characterize the class (E) of cyclicly connected planar graphs G such that there is a map of G on the sphere in which all homeomorphisms T of G to itself can be extended to the sphere. Terminology: a *fixed-point* of G under T is either a vertex p with $Tp=p$, an edge μ with $T(\mu)=\mu$ or a circuit with $T(C)=C$ and with no fixed-point edges or vertices of T on C. If $G=H_1+H_2$, where the non-void subgraphs H_1 and H_2 have no edges and only two vertices p and q in common, where neither H_1 nor H_2 is a chain, and where one of H_1, H_2 is not disconnected by the removal of p and q, then H_1 is a side of any circuit $C \subset H_1$. Theorem: G belongs to the class (E) if and only if every T of order 2 has neither (i) a circuit D of fixed edges and vertices and a fixed point not on D, nor (ii) a fixed-point circuit C and three fixed points such that any two lie on a side of the third, whenever the third is a circuit. (Received February 23, 1937.)

132. Dr. H. M. MacNeille: *Infinite symmetric differences.* I, *Absolute convergence.* Preliminary report.

Consider a Boolean ring, K, of point sets, $a, b, c, \cdots$, where $a-b$, the ring addition, represents all points in either a or b but not in both, and ab, the ring multiplication, represents all points in both a and b. A series, $D_{i=0}^{\infty}a_i=a_0-a_1-a_2-\cdots$, is absolutely convergent if $\prod_{n=0}^{\infty}\sum_{i=n} a_i=0$, and represents all points which occur an odd number of times in the sequence $\{a_i\}$. The set, L, of absolutely convergent series in K is a Boolean ring containing K as a subring. In the notation of Hausdorff, *Mengenlehre* (de Gruyter, Leipzig, 1927), §18, L contains K_σ and K_δ and is contained in the common part of $K_{\sigma\delta}$ and $K_{\delta\sigma}$. Every element of L can be expressed in the form $\sum_{j=0}^{\infty}\prod_{i=0}^{\infty}a_{ij}$ and so that $\sum_{j=0}^{\infty}\prod_{i=0}^{\infty}a_{ij}=\prod_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{ij}$. Whenever $\sum_{j=0}^{\infty}\prod_{i=0}^{\infty}a_{ij}=\prod_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{ij}$, then $\sum_{j=0}^{\infty}\prod_{i=0}^{\infty}a_{ij}=\lim \sum_{j=0}^{i}a_{ij}$ and the limit exists. Furthermore, whenever $\lim a_i$ exists, the point set so represented is in L, thus completing the cycle. The extension from K to L can be employed to construct the smallest completely additive system containing K, by methods analogous to those of Hausdorff. (Received March 4, 1937.)

133. Mr. W. G. Madow: *Fundamental theorems of comparative statistical analysis.*

It is the purpose of this paper to develop the probability distributions which are necessary for the solution of problems which arise when the relative effects of several factors influencing an experiment are investigated. The fundamental distributions are those which generalize the theorem of Poincaré on the probability that at least one of several events occur. There are evaluated the probability that a definite j of k events occur; the probability that exactly j of k events occur; and the probability that at least j of k events occur. These results are extended to events of different kinds, the final theorem of this group being the evaluation of the probability that a definite j_ν of k_ν events of kind ν occur, $\nu=1, \cdots, g-1$, that exactly j_ν of k_ν events of kind ν occur, $\nu=g, \cdots, h-1$, and that at least j_ν of k_ν events occur, $\nu=h, \cdots, p$. Events satisfying a relation of implication are then considered, the final theorem being a generalization of a theorem in which is evaluated the probability that at least j_1 events of kind 1 and at least j_2, $j_2 \geqq j_1$, events of kind 2 occur, the occurrence of event μ of kind 1 implying the occurrence of events μ of kind 2. The proofs are based on the Kolmogoroff axioms. No assumptions of independence are made. (Received March 4, 1937.)

134. Mr. W. G. Madow: *The generalized analysis of variance.*

In a paper presented to the Institute of Mathematical Statistics the author has stated several generalizations of the Fisher-Cochran theorem. In this paper these generalizations are used in order to extend the theory of the generalized analysis of variance. Let a_i, $i=1, \cdots, p$, be the generalized variance consisting of the first i rows and columns of the p-rowed generalized variance of variables which have a joint normal distribution. There are derived the joint distributions of $a_1, a_2, \cdots, a_p$ and of similarly formed generalized correlation ratios. Various distributions related to sets of observations are obtained. These distributions include some which may be used in testing hypotheses concerning relations between sets of variables. To extend the results concerning sets of variables to cases of known relations between the sets there is obtained a theorem by which, under certain fairly general conditions, distributions derived by assuming a particular probability density are immediately transformed into distributions of the same functions of variables which have some other density. (Received March 4, 1937.)

135. Professors N. H. McCoy and Deane Montgomery: *A representation of generalized Boolean rings.*

Stone has recently shown (Transactions of this Society, vol. 40, pp. 37–111) that every Boolean ring is isomorphic to a ring of subclasses of some class, addition of classes being carried out modulo two. An equivalent formulation of this theorem is to the effect that every Boolean ring is isomorphic to a subring of a direct sum of rings F_2 (F_p being used to denote the ring of integers reduced modulo p). The authors give here a simple proof of this theorem in a somewhat more general case. It is proved, namely, that if p is any prime and if R_p is a ring such that $a^p=a$, $pa=0$, for all a, then R_p is isomorphic to a subring of a direct sum of rings F_p. The proof is based on a theorem stating that a necessary and sufficient condition that a ring R be isomorphic to a subring of a direct sum of rings S is that for every $a \neq 0$ in R there is a homomorphism f taking R into a subring of S such that $f(a)\neq 0$. (Received February 18, 1937.)

136. Professor Rufus Oldenburger: *Real symmetric trilinear forms.*

In this paper there is obtained a complete set of canonical binary symmetric trilinear forms for the class of forms which can be written as $a_{ijk}x_iy_jz_k$, the field of reals, and non-singular linear transformations on x_i, y_j. There are in all 6 canonical forms, characterized by invariant ranks and a simple invariant property of generalized invariant factors of 3-way matrices. The problem is treated by associating with each binary symmetric trilinear form a quadratic form, which is never positive definite. Reduction of this quadratic form to canonical form leads to a canonical form of the associated trilinear form. For a complete bibliography of papers on trilinear forms the reader is referred to an article by the author in this Bulletin, vol. 38 (1932), pp. 385–387. (Received March 5, 1937.)

137. Professor Oystein Ore: *Extension of a theorem of Markoff.*

The well known theorem of Markoff states that if a polynomial $f(x)$ of degree n is bounded so that $|f(x)| \leqq M$ in the interval $(-1, 1)$, then, in the same interval, $|f'(x)| \leqq n^2M$. It is shown that this theorem may be given such a form that it holds for all differentiable functions. (Received March 4, 1937.)

138. Mr. B. J. Pettis: *On integration in vector spaces.*

A function $f(s)$ defined to a Banach space B from an abstract space S having a measure function $\alpha(E)$ is said to be "integrable over E" if there exists an x_E in B such that $T(x_E)=\int_E T(f(s))d\alpha$ for every T in $\bar{B}$; it is "integrable" if it is integrable over every measurable set. The integral is completely additive and absolutely continuous and includes all other known definitions except the most recent ones of Dunford (this Bulletin, vol. 43, No. 1 (1937)) and G. Birkhoff (Annals of Mathematics, vol. 38, No. 1 (1937)). For separable spaces this definition reduces to one given by Birkhoff (Transactions of this Society, vol. 38, No. 2 (1935)). (Received March 4, 1937.)

139. Mr. W. T. Puckett: *A theorem concerning homologies in a compact space.*

Let M be a compact set imbedded in the n-dimensional euclidean space and let M^r be any closed subset which contains all the rth order cyclic elements of M (see G. T. Whyburn, American Journal of Mathematics, vol. 56 (1934), pp. 133–146). This paper proves: If γ^{r+1} is any complete $(r+1)$-dimensional cycle in M, then there exists a complete $(r+1)$-dimensional cycle γ^{*r+1} in M^r such $\gamma^{*r+1} \backsim \gamma^{r+1}$ in M. Thus the $(r+1)$-dimensional Betti group (mod 2) of M^r is isomorphic with the corresponding group of M. This is an extension of a theorem by S. Eilenberg (Fundamenta Mathematicae, vol. 24 (1935), pp. 151–155). (Received March 2, 1937.)

140. Professor C. J. Rees: *Elliptical moments.*

In this paper the author studies elliptical moments, that is, moments with weight functions $[(1-x^2)(1-k^2x^2)]^{-1/2}$, $[(1-k^2x^2)/(1-x^2)]^{1/2}$, $(1+nx^2)^{-1} \cdot [(1-x^2)(1-k^2x^2)^{-1/2}$, where $0<k<1$. Difference equations, differential equations, and other relations satisfied by these moments are considered and new methods are employed in arriving at those particular results which are not new. Consideration is given to the coefficients in the expressions for an arbitrary elliptical moment in terms of the first two moments, which are orthogonal polynomials. Furthermore, the study is related to that of the elliptical polynomials $\Phi_n[x; -1, 1; (1-x^2)^{-1/2}(1-k^2x^2)^{-1/2}]$. (Received March 3, 1937.)

141. Mr. L. B. Robinson: *On complete systems of tensors.*

The author has computed some simple cases of relative covariant tensors of weight 2 associated with a system of linear homogeneous ordinary differential equations and has developed a method which will solve very general cases by quadratures. The tensors are found as integrals of a system of Riquier (see page 502 of his book). (Received March 4, 1937.)

142. Dr. J. B. Rosser: *An improvement of Brun's method in number theory.* Preliminary report.

For x of a large order relative to $k(x>\exp(k^{1/2})$, for instance) $\pi(x; k, l)$, the number of primes $<x$ in an arithmetic sequence with difference k and first term l, can be evaluated by a study of the Dirichlet L-series. By the Brun method one can get an upper bound for $\pi(x; k, l)$ for much smaller x. For instance, by the improved Brun method one can show that for large k, $n>1000$, $x>k^n$, and $(k, l)=1$, $\pi(x; k, l)<[(2.0001(1+1/n)+\epsilon)x]/[\phi(k) \log x]$. This is derived by computing θ_2 to four-place accuracy in the following theorem. There are constants θ_1 and θ_2 such that for $n>1000$ and positive δ there is a $k_0(n, \delta)$ such that, if $k>k_0(n, \delta)$ and $x>k^n$ and $(k, l)=1$, the number of positive integers $<x$ and $\equiv l (\text{mod } k)$ whose prime factors are all $>x^{1/3}$ is comprised between $[((1-1/n)\theta_1-\delta)x]/[\phi(k) \log x]$ and $[((1+1/n)\theta_2+\delta)x]/[\phi(k) \log x]$. The constants θ_1 and θ_2 are given by infinite series which have not been precisely evaluated as yet. However there is strong reason to believe that $\theta_1=\log 4$ and $\theta_2=2$, and this conjecture is further supported by the fact that actual computation gives $|\theta_1 - \log 4| < 10^{-4}$ and $|\theta_2 - 2| < 10^{-4}$. (Received March 5, 1937.)

143. Dr. I. J. Schoenberg: *On a certain restricted class of convex functions.* Preliminary report.

A real-valued point function $f(P)$ defined throughout euclidean space $\mathfrak{R}_n$ is convex if (1) $\overline{PR}f(Q)+\overline{PQ}f(R)-\overline{QR}f(P)\geqq 0$ whenever P is on the straight segment QR. Here one is concerned with functions $f(P)$ for which (1) holds for any three points P, Q, R of $\mathfrak{R}_n$, and such functions are called *hyperconvex.* By Ptolemy's inequality of elementary geometry, the functions of the form (2) $f(P)=c+\int \overline{PM}d\phi(M)$ $(c\geqq 0)$, where $d\phi(M)$ is a positive distribution in $\mathfrak{R}_n$ for which the Stieltjes integral over $\mathfrak{R}_n$ exists, are hyperconvex functions. For the simplest case of $n=1$, this formula (1) is found to give the most general hyperconvex function. This is no longer true for $n\geqq 2$. (Received March 5, 1937.)

144. Dr. G. E. Schweigert: *Concerning homeomorphisms and interior transformations.*

Given a compact metric space A and a periodic homeomorphism $H(A)=A$, there exists an interior transformation $T(A)=B$ where each point of B (or each point b) consists of a point x of A together with its images under h. The analytic character of these transformations as shown by previous writers is reviewed in the light of these results. (Received March 9, 1937.)

145. Professor I. M. Sheffer: *Concerning Appell sets and linear functional equations.*

Let $\{P_n\}$ be an Appell set with a generating function $L(t)$ that is of exponential type. An asymptotic relation is obtained for $\{P_n\}$, in terms of the singularities of a function associated with $L(t)$. The level curves of the set $\{P_n\}$ are shown to be simple, closed, convex curves. Application is made to the problem of finding a local solution of related linear functional equations. (Received March 3, 1937.)

146. Mr. Max Shiffman: *Minimal surfaces with prescribed contours not having least area.*

The solution of Plateau's problem in the case of more contours or higher topological structure of the required surface depends on certain inequalities as sufficient conditions. These inequalities mean that the prescribed topological structure gives a lower bound for the area smaller than does a lower topological structure or a degenerated surface. There are, however, as the case of minimal surfaces of revolution shows, cases when Plateau's problem is solvable although these conditions are not satisfied and the solution does not give the least area. For such cases, Courant's existence proof can be applied without modifications. The only difference is that instead of the original inequality certain other inequalities are substituted as sufficient conditions. These inequalities express the condition that on the surfaces admissible in the variational problem certain families of curves shall have a diameter above a fixed positive number. (Received March 4, 1937.)

147. Professor J. A. Shohat: *On the convergence properties of the Lagrange interpolation formula based on the zeros of orthogonal polynomials.*

Let $\psi(x)$ be monotone and non-decreasing in (a, b) with all properties attributed to it in the theory of orthogonal polynomials. If the integral from a to b of $f^2(x)d\psi(x)$ exists, the Lagrange interpolational polynomial for $f(x)$ based on the zeros of the orthogonal polynomials corresponding to $\psi(x)$ and (a, b) finite, *converges in the mean* to $f(x)$. This holds for an infinite interval for some classes of orthogonal polynomials. The proof utilizes the theory of mechanical quadratures for orthogonal polynomials. (Received February 26, 1937.)

148. Professor L. L. Silverman: *On a general class of consistent definitions of summability.*

The matrix corresponding to Nörlund's definition of summability is given in terms of an auxiliary sequence of numbers. In this generalization of Nörlund's definition, the matrix is given in terms of two auxiliary sequences. Such generalized matrices are accordingly called *bisequential.* When the elements of one or the other auxiliary sequence are all unity the matrix reduces either to Nörlund's or to that corresponding to the typical means of M. Riesz. Two bisequential matrices are *conjugate* if the two auxiliary sequences are interchanged. The matrices corresponding to the definitions of Nörlund and Riesz are conjugate. If the two auxiliary sequences are identical the matrix is *self-conjugate.* The matrix corresponding to Euler's definition of summability appears as a special self-conjugate matrix. Corresponding to every Cesàro matrix of order r there exists an equivalent matrix which is self-conjugate. Regarding these bisequential matrices theorems are obtained on their mutual consistency, on the permissibility of altering a finite number of terms of a summable series, on the product of summable series, and on the generalization of Abel's theorem on power series. (Received January 29, 1937.)

149. Professor Virgil Snyder and Evelyn C. Rusk: *A series of involutorial Cremona transformations in S_n belonging multiply to a non-linear line complex.*

A pencil of quadric primal is given, and a linear space S_{n-2} defined by two linear equations $(ax)=0$, $(bx)=0$. The coefficients a and b are polynomials in a parameter. Between this parameter and that of the pencil is an $(n-1, k)$ involution. A point defines a quadric primal through it and $n-1$ director spaces. The line through the point meeting all the director spaces pierces the quadric again in a residual point, the conjugate of the given one in the Cremona involution. A line joining a point to its conjugate contains k such pairs of conjugates. (Received February 25, 1937.)

150. Professor C. T. Sullivan: *Scrolls whose asymptotic curves are cubics.*

In this paper the equations of all scrolls possessing the property specified are given explicitly. The constants occurring in the equations are classified in such a way that the equations of each of the possible types may be written down in advance. These types include cubic, quartic, quintic, and sextic scrolls. With the exception of one type of quartic, the scrolls have been placed in the appropriate classes as enumerated by Edge. The exceptional quartic mentioned appears to have escaped detection in the classical algebraic classifications. (Received February 13, 1937.)

151. Professor J. L. Synge: *The absolute optical instrument.*

An absolute optical instrument is one in which there is an exact image of each object-point, light being supposed to obey Fermat's principle. Carathéodory (Sitzungsberichte der Bayerischen Akademie der Wissenschaften, mathematisch-naturwissenschaftliche Abteilung, 1926, pp. 1–18) gave a very general theorem to the effect that the optical length of any object-curve is equal to that of its image. It is now shown that his theorem cannot be true in the general form stated, and that the proof given is not valid on the basis of the hypotheses. A systematic method is developed, based on Hamilton's medium-equation, and the equality of optical lengths is established for the case of isotropic terminal media. This method appears to possess advantages over the methods previously employed in the restricted case of isotropic homogeneous terminal media. In the case of homogeneous crystalline terminal media, it is shown that the principal velocities in the two media must be proportional if the instrument is absolute; the equality of optical lengths is established in this case also. (Received March 4, 1937.)

152. Professor J. L. Walsh and Dr. W. E. Sewell: *A note on the relation between degree of polynomial approximation and continuity in the complex domain.*

An application of conformal mapping to results of de la Vallée-Poussin on trigonometric approximation yields the following theorems: 1. Let C be an analytic Jordan curve in the z-plane and let $f(z)$ be defined in $\overline{C}$, the closed limited set bounded by C. If for every n, $n=1, 2, \cdots$, there exists a polynomial $P_n(z)$ of degree n in z such that $|f(z)-P_n(z)| \leqq M/n^{p+\alpha}$, z in $\overline{C}$, $0<\alpha\leqq 1$, where M is a constant independent of n and z, and p is a positive integer or zero, then $f(z)$ is analytic in C, continuous in $\overline{C}$, and $|f^{(p)}(z_1)-f^{(p)}(z_2)| \leqq L|z_1-z_2|^{\alpha}\cdot |\log|z_1-z_2||^{\beta}$, z_1, z_2 on C, where $\beta=0$ if $\alpha<1$, and $\beta=1$ if $\alpha=1$, L is a constant independent of z_1 and z_2, and $f^{(p)}(z)$, $p\geqq 0$, is the pth derivative (in the one-dimensional sense) of $f(z)$ on C, $f^{(0)}(z)\equiv f(z)$. 2. Let E, with boundary C, be a closed limited set in the z-plane whose complement K is connected and regular, and let C_R, $R>1$, be an equipotential locus which consists of a finite number of mutually exterior analytic Jordan curves. If (using the notation above) one has $|f(z)-P_n(z)| \leqq M/(n^{p+\alpha+1}R^n)$, z in E, $n=1, 2, \cdots$, then $f(z)$ is analytic interior to C_R, continuous in $\overline{C}_R$, and $|f^{(p)}(z_1)-f^{(p)}(z_2)| \leqq L|z_1-z_2|^{\alpha}\cdot|\log|z_1-z_2||^{\beta}$, z_1, z_2 on C_R. (Received February 16, 1937.)

153. Professor Morgan Ward: *A note on residuals in commutative rings.*

If R and S are ideals of a commutative ring O with unit element, the residual $R:S$ is an ideal with the properties $R \supset (R:S)S$; if $R\supset TS$ then $R:S\supset T$.

Define the quotient $Q=R/S$ whenever it exists by $R=QS$; if $R=YS$, then $Q \supset Y$. With this definition, R/R, R/O exist and RS/R exists. The author proves here that whenever one of the quotients $R/(R, S)$ or $[R, S]/R$ exists, it equals the residual $R:S$. Here (R, S) and $[R, S]$ denote the union and cross-cut of R and S. (Received March 4, 1937.)

154. Mr. G. C. Watson: *Almost monotone transformations.*

A single-valued, continuous transformation, $T(A)=B$, is said to be almost monotone provided that the inverses of all save possibly a finite number of the points of B are connected. In this paper, it is shown that a necessary and sufficient condition that a continuum be an even linear graph plus a null sequence of simple closed curves at each vertex of the graph is that it be the image of the circle under an almost monotone transformation. Furthermore, the property of being an even linear graph plus a null sequence of simple closed curves at each vertex is invariant under almost monotone transformations. (Received February 8, 1937.)

155. Professor Hassler Whitney: *Analytic coordinate systems and arcs in a manifold.*

The author considers some fundamental properties of an analytic manifold M, under the assumption that it can be imbedded analytically in a euclidean space. If N is an analytic submanifold of M in "regular position" in M (compare Whitney, Annals of Mathematics, 1936, p. 867), it may be imbedded in a family of homeomorphic analytic manifolds filling out an open subset of M. This is always true if N is of the nature of a cube of some dimension; in this case the family of manifolds becomes a coordinate system containing N. A finite set of points may be joined by a differentiable arc; this arc may be imbedded in a coordinate system, and an analytic arc is then found joining the points. (T. Y. Thomas, Annals of Mathematics, 1937, has proved this under weaker assumptions.) Any analytic simple closed curve bounding a differentiable surface element also bounds an analytic one. The methods employed are those of Whitney, Annals of Mathematics, 1936, pp. 645–680. (Received March 5, 1937.)

156. Professor Hassler Whitney: *On cocycles and products in a complex.*

The invariance theorems for homology groups in a complex and intersections in a manifold are well known. The object of this paper is to prove the same for cohomology groups and products in a complex. The treatment is along simple, classical lines. The definition of products was discovered independently by Čech and the author (see Čech, Annals of Mathematics, 1936); the author extends the work of Čech by finding all products such that two cells multiply into a chain lying in the stars of the cells, and so that $\delta(A^r \smile B^s) = \delta A^r \smile B^s + (-1)^r A^r \smile \delta B^s$, ($\delta$ = coboundary). The complex need not be simplicial. For each product $\smile$, $\delta \smile \delta = \gamma\delta$ for some integer γ. The important case is $\gamma = 1$. The products are determined in product complexes, and some mapping theorems are proved. For the "dual" f of a simplicial map, $f(A \smile B)$

$=f(A)\smile f(B)$. (This was discovered by the author in 1935; see also Freudenthal, Amsterdam Akademie van Wetenschappen, 1937, pp. 2–8.) (Received March 5, 1937.)

157. Professor G. T. Whyburn: *Interior transformations on compact sets.*

Let A be compact and suppose $T(A)=B$ is an interior transformation such that for each $y\epsilon B$, $T^{-1}(y)$ is totally disconnected. It is shown that if pq is any simple arc in B and $p_0\epsilon T^{-1}(p)$, there exists a simple arc p_0q_0 which maps *topologically* onto pq under T; if J is any simple closed curve in B, either there exists a simple closed curve C in A mapping onto J under T and such that $T(C)=J$ is topologically equivalent to the transformation $w=z^k$ on $|z|=1$ (k an integer) or there exists an open curve L in A such that $T(L)=J$ is equivalent to the transformation $x=\cos t$, $y=\sin t$ on $-\infty<t<\infty$; if K is a locally connected continuum in B, each component of $T^{-1}(K)$ maps *interiorly* onto K under T. If A is locally connected and lies on a two-dimensional manifold, then for each locally connected continuum K in B, $T^{-1}(K)$ is locally connected. By using the factor theorem ($T=T_2T_1$, T_1 monotone, $T_2^{-1}(y)$ totally disconnected for each $y\epsilon B$) similar conclusions are obtained without the restriction on the sets $T^{-1}(y)$. (Received February 19, 1937.)

158. Professor G. T. Whyburn: *On the mapping of Betti groups under interior transformations.*

An example is given of a compact continuum A and an interior transformation $f(A)=B$ such that the one dimensional Betti group of A, even relative to the rational coefficient field, does not map homomorphically into the corresponding group of B. (The example $w=z^2$ on $|z|=1$ shows this for the integral coefficient domain.) However, if A is a graph, B is a graph and there exist subdivisions of A and B into complex K_a and K_b respectively so that $f(K_a)=K_b$ is simplicial and f maps each edge in K_a topologically into an edge of K_b; furthermore the Betti groups of K_a, relative to the rational field, map homomorphically into those of K_b so that $p^1(B)\leqq p^1(A)$. This same inequality is proven under the less restrictive assumption that A is a compact one-dimensional locally connected continuum, and it is established for graphs A without endpoints undergoing any continuous transformation f which increases the order of no point. (Received February 19, 1937.)

159. Dr. L. R. Wilcox: *Generalizations of affine geometry.*

In classical geometry the operation (A) of deleting a hyperplane from a projective space gives rise to an affine space. The inverse operation (B) consists in adding ideal elements to an affine space to obtain a projective space. In this paper a lattice-theoretic formulation of the operation (A) is given, and a generalization of it to the continuous geometries of von Neumann is found. This procedure leads to a natural definition of affine lattices; properties of these lattices are investigated. A study is made of an extension to affine lattices of a formulation of (B) due to Veblen (Transactions of this Society, vol. 5 (1904), pp. 343–384). (Received March 5, 1937.)

160. Professor S. S. Wilks: *The large-sample distribution of the likelihood ratio for compositie hypotheses.*

Let $f(x, \theta_1, \theta_2, \cdots, \theta_r)$ be a probability function of x defined for all points $P(\theta_1, \theta_2, \cdots, \theta_r)$ in an r-dimensional region R_Ω of the θ's. Let R_ω be an m-dimensional region $(m<r)$ which is contained in R_Ω. Let π_Ω and π_ω be the least upper bounds in R_Ω and R_ω of the likelihood of a sample O_n of n individuals. The likelihood ratio for testing from O_n the composite hypothesis $H[\Omega; \omega]$ that P is in R_ω is given by $\lambda = \pi_\omega/\pi_\Omega$. Under the assumption that optimum estimates of the θ's exist, it is shown that when n is large the quantity $-2 \log \lambda$ is approximately distributed in a X^2 distribution with $r-m$ degrees of freedom. The results hold for probability functions of several variables. Applications are made to certain important hypotheses which arise in statistics, such as independence in contingency tables, the mutual independence of several sets of variates, and hypotheses relating to means, variances, and covariances in the problem of generalized analysis of variance. (Received March 4, 1937.)

161. Dr. John Williamson: *The conjunctive equivalence of pencils of hermitian and anti-hermitian matrices.*

Let K be a commutative field of characteristic zero and let K_1 be a quadratic adjunction field of K. If A is a matrix with elements in K_1, A^* is defined to be the conjugate transposed of A. Two matrices A and A_1 with elements in K_1 are said to be conjunctively equivalent if there exists a non-singular matrix P with elements in K_1 such that $P^*AP = A_1$. Necessary and sufficient conditions for the conjunctive equivalence of two matrix pencils $rA+sB$ and rA_1+sB_1, where $A = \epsilon A^*$, $B = \delta B^*$, and ϵ, $\delta = \pm 1$, are determined. These conditions are (1) that the Kronecker minimal indices of the two pencils be the same, (2) that the invariant factors of the two pencils be the same, and (3) that certain diagonal matrices be equivalent over fields of K. Corresponding results are obtained when all matrices are restricted to be matrices over K. In particular it is shown that (1) and (2) are necessary and sufficient conditions for two pencils of skew-symmetric matrices to be congruent. (Received February 19, 1937.)

162. Professor B. C. Wong: *Projections of algebraic varieties.* Preliminary report.

The projective characteristics: the order n; the various classes $m_1, m_2, \cdots, m_k$; the orders $b_1, b_2, \cdots, b_k$ of the double varieties on the various general projections; and the orders $j_1, j_2, \cdots, j_k$ of the loci of pinch points on the various projections of a given k-variety V_k in an r-space are supposed known. The projection, V_k', of V_k from a center of projection having ν points in common with V_k upon a lower space has its own corresponding characteristics n'; $m_1', \cdots, m_k'$; $b_1', \cdots, b_k'$; $j_1', \cdots, j_k'$. The purpose of this paper is to determine the relations between the characteristics of V_k and those of V_k'. Formulas for the maximum values of b_k for a V_k in S_{2k} where V_k is the locus of ∞^h $(k-h)$-spaces for $0<h\leqq k$ are also obtained. (Received March 2, 1937.)

163. Dr. E. T. Welmers: *Set functions and measurability conditions.*

A generalization of the Carathéodory condition for measurable sets is made in which the set functions need not be defined for the entire class of sets and need not satisfy the usual additivity and value space postulates; the requirement that the condition be satisfied for every subset of the space is also weakened. Some sufficient conditions are obtained that the class of measurable sets form a ring and that the function have additive properties. When the value space is linearly ordered, greatest lower bound and least upper bound extensions by means of measurable sets are possible; certain properties of the Carathéodory theory result, depending on the nature of the extension, and for one extension essentially the same properties are obtained. An application of this extension is made to a set function defined on an arbitrary group and a measure similar to the Haar and von Neumann group measures is thereby defined. (Received February 26, 1937.)

164. Dr. W. C. Randels: *On the order of the partial sums of Fourier series.*

An example is given to show that the estimate $s_n = o(n)$ for the partial sums of a Fourier series cannot be improved on. Similar results are given for the Cesàro transforms. (Received February 26, 1937.)

165. Mr. W. A. Dwyer: *On certain fundamental identities due to Uspensky.*

In this paper the author obtains a theta-function identity which, by the principle of paraphrase of E. T. Bell (Transactions of this Society, vol. 22 (1921), pp. 1–36, 198–220) establishes an identity involving incomplete numerical functions in three variables. Two fundamental identities discovered by Uspensky (Bulletin de l'Academie des Sciences de l'U.S.S.R , series 6, vol. 19–20 (1925–26); this Bulletin, vol. 36 (1930), pp. 743–755), for which he gave purely arithmetic proof, result as special cases of the identity obtained. The analytical procedure provides a systematic method of deriving other formulas of this kind. This paper will appear shortly in the American Journal of Mathematics. (Received February 26, 1937.)

166. Mr. W. A. Dwyer: *On certain pseudo-periodic functions.*

In the preceding paper (abstract 43-3-165), the author obtained a theta function identity involving a term of the form $\theta_\alpha(x+y+z)\phi_{abc}(u, v)$, where $\phi_{abc}(u, v)$ are the doubly periodic functions of the second kind. The sixty-four products of this type give rise (through the method given by M. A. Basoco, American Journal of Mathematics, vol. 54 (1932), pp. 242–253) to as many identities, which may be grouped into eight sets of eight each. The members of a set may be obtained from a fundamental identity of the set by simple transformations. By application of Bell's principle of paraphrase, these theta identities lead to formulas similar to those obtained by Uspensky (see abstract 43-3-165). Some of the arithmetical consequences of these paraphrases are now being studied. (Received February 26, 1937.)

167. Dr. Nathan Jacobson (National Research Fellow). *A note on non-associative algebras.*

Let $\mathfrak{R}$ be an arbitrary algebra (not necessarily associative) with a finite basis over a commutative field Φ, and let $\mathfrak{a}$ be the enveloping algebra of the linear transformations $x \rightarrow ax$ and $x \rightarrow xa$ determined by the elements a of $\mathfrak{R}$. Relations between the structure of $\mathfrak{R}$ and that of $\mathfrak{a}$ are discussed. In particular it is shown that if $\mathfrak{R}$ is simple $\mathfrak{a} \cong P_n$ the complete matrix algebra of degree n over a finite algebraic extension P of Φ. $\mathfrak{R}$ may be regarded as an algebra over P, and when this is done $\mathfrak{R}$ becomes normal simple; that is, remains simple when P is extended to its algebraic closure. If S is an automorphism of $\mathfrak{R}$ over Φ it is shown that S induces an automorphism in P and $(x\xi)^S = x^S \xi^S$ for $\xi \epsilon P$. This essentially reduces the problem of determining the automorphisms of $\mathfrak{R}$ over Φ to that of determining the automorphisms of $\mathfrak{R}$ over P. If $\mathfrak{R}$ is specialized to be associative the results give an elementary proof of Brauer's theorem that the direct product of a normal simple algebra and its reciprocal is matric. (Received March 1, 1937.)

168. Dr. Nathan Jacobson (National Research Fellow). *p-algebras of exponent p.*

Let A be a normal simple algebra of order p^{2m} over a field F of characteristic p, and suppose A has a maximal inseparable subfield $C = F(c_1, \cdots, c_m)$, $c_i^p = \gamma_i \epsilon F$. It is known that A is cyclic. In this note a new generation of A is obtained based on a derivation in C, and a condition in terms of this generation that A be a complete matrix algebra is derived. In the special case $m=1$, A is generated by c such that $c^p = \gamma$, and a second element d such that $d^p = \delta$ and $cd - dc = 1$. This symmetric generation and the condition that A be matric yield a reciprocity law for arbitrary imperfect fields. (Received March 1, 1937.)

169. Dr. H. H. Goldstine: *On extensions of linear and bilinear transformations.*

It is known that in general it is not possible to extend a linear and continuous transformation. In fact, even if the domain and contra-domain of such an operation are separable, Banach and Mazur have shown that an extension does not necessarily exist. However the author shows that a linear and continuous transformation between a linear subset of a modular space and a similar subset of another modular space can always be extended in such a manner that the extension is linear and continuous on the entire modular space, has the same modulus as the original operation, and coincides with the first transformation on its domain of definition. Then since the classical Hilbert space is a modular space and since the former is isomorphic with any arbitrary abstract Hilbert space, the theorem is true in Hilbert spaces. It is also shown that bilinear and continuous functionals have a bilinear and continuous extension. (Received March 2, 1937.)

170. Professor H. S. Wall: *On continued fractions representing constants.*

This paper is concerned with the determination of sub-spaces Σ of the space

S of points $x=(x_0, x_1, x_2, \cdots)$ in which the continued fraction $x_0+K(x_n/1)$ has a constant value. One method is to suppose that the x_n depend upon parameters $t_1, t_2, t_3, \cdots$ in this fashion: for some integer $p>1$, $x_{kp+1}, x_{kp+2}, x_{kp+3}, \cdots, x_{kp+p}$ depend upon $t_{k+1}, t_{k+2}, \cdots, t_{k+q}$ (q constant, $k=0, 1, 2, \cdots$). A second method is to employ certain convergence preserving transformations which transform the x_n into themselves from and after some value of n, and in a properly chosen sub-space of S. As an example, let $x_0=0$, $x_1=(1+t_1)^{1/2}$, $x_{2n}=t_n$, $x_{2n+1}=[(1+t_n)(1+t_{n+1})]^{1/2}$, $n=1, 2, 3, \cdots$. Then if the continued fraction converges its value is either $+1$ or -1. Here $t_1, t_2, t_3, \cdots$ are arbitrary complex numbers. (Received March 2, 1937.)

171. Mr. R. M. Thrall: *Metabelian groups and trilinear forms.*

This paper is concerned with metabelian groups conformal with the abelian group of order p^n and type $1, 1, 1, \cdots$ and trilinear forms belonging to a finite field, and is an attack on some of the problems presented by Brahana (Duke Mathematical Journal, vol. 1 (1935), pp. 185–197). First the properties of the groups are interpreted in the forms, next there is given a general method of classification of trilinear forms under linear transformations and the three sets of variables taken separately. This method involves representation of n-ary m-ics as m-rowed determinants with linear elements. The application of this theory to the classification of ternary trilinear forms follows and the results are interpreted in group theoretic terms. Incidental results include a new proof of the existence of the "imaginary" ternary cubic form and the exhibition of metabelian groups with an arbitrarily large number of non-commutators in the commutator subgroup. (Received March 3, 1937.)

172. Professor M. H. Ingraham and Dr. M. C. Wolf: *Convergence of a sequence of linear transformations.*

This paper treats the problem of finding sufficient conditions that a sequence of linear transformations are such that the circle, sphere, or hypersphere approaches the origin uniformly under the application of the transformations of the sequence. The two-dimensional case with real characteristic values and directions is discussed in detail. The problems involved arise in studying the nature of the neighborhood of an invariant point of a transformation under a sequence of transformations which differ only slightly from the given transformation. Such a study seems necessary for a mathematical approach to a theory of genetic stability where environmental conditions vary sufficiently to make the evolutionary effect change a small amount from generation to generation. (Received March 3, 1937.)

173. Professor M. H. Ingraham: *Note on determinants.*

If A is a square matrix of square matrices A_{ij} which are mutually commutative, then the determinant of A is the determinant of the matrix obtained by taking the determinant of A and treating the A_{ij} as individual elements. This is particularly useful when the A_{ij} are all polynomials in a single matrix. Let $A_{ij}=g_{ij}(B)$, then if $|(g_{ij}(\lambda)|=g(\lambda)$, $|A|=|g(B)|$. (Received March 3, 1937.)

174. Professor H. P. Evans and Dr. S. C. Kleene: *On the postulational basis of probability*. Preliminary report.

This paper deals with the formulation of a set of postulates, the purpose of which is to fit the various concepts of probability into a single logical structure based on the calculus of propositions. Certain concepts introduced by Reichenbach and by Kolmogoroff in their recent books are merged in this treatment. Three axioms yield the addition and multiplication theorems for probabilities and the scale of probability measurement, and lead to the definitions of mutually exclusive and independent events and of conditional probabilities. Some further axioms are necessary to extend the theory into the domain of analysis using probability functions, and to connect the theory with the relative frequency theory of statistics. (Received March 3, 1937.)

175. Professor R. P. Baker: *Space fillers in the plane*.

If it is postulated only that a set of congruent polygons fill the plane indefinitely without gaps, the list of cases includes those which have been made hitherto with further restrictions concerning groups, lattices, equivalence of aspects, and so on, but is much larger. The cardinal number is that of the continuum. This arises either from "variable" angles or from enumerable choices of orientation as the packing proceeds. With the added postulate that each polygon when packed has the same specification of vertices, there are ten cases. If (4.3.4.3.3) is interpreted as a pentagon with vertices $v_4v_3v_4v_3v_3$, these are (3.3.3.3.3.3); (6.3.3.3.3), (4.4.3.3.3), (4.3.4.3.3); (6.4.3.4), (6.3.6.3), (4.4.4.4); (12.12.3), (8.8.4), (6.6.6). Several of these have different types of solutions and moreover "mixed" and "arbitrary" cases occur. Fifty diagrams cover the general classification and are given. (Received March 4, 1937.)

176. Mr. M. F. Smiley: *Discontinuous solutions for the problem of Bolza in parametric form*.

Graves has treated minimizing arcs with corners, the so-called discontinuous solutions, for the parametric problem of the calculus of variations in n-space with fixed end-points and no side differential equations (American Journal of Mathematics, vol. 52 (1930), pp. 1–28). Some of his results were generalized by Hefner to the problem of Lagrange with fixed end-points (*Contributions to the Calculus of Variations* (1931–2), The University of Chicago Press, pp. 95–130). More recently Reid has discussed discontinuous solutions for the general non-parametric problem of Mayer (American Journal of Mathematics, vol. 57 (1935), pp. 69–93). The principal object of the present paper is to give sufficient conditions for a discontinuous solution of the general problem of Bolza in parametric form. Several effective Jacobi conditions are given. The sufficiency proof is direct and is made by a method recently used by Hestenes for the case of extremal arcs in the non-parametric problem of Bolza (abstract 42-11-450). (Received March 4, 1937.)

177. Professor Dunham Jackson: *Note on properties of invariance of orthogonal polynomials in a complex variable*.

If a system of orthogonal polynomials in two real variables is constructed for a region and a weight function both of which are invariant under a linear transformation of the variables, this fact is associated with certain properties of transformation of the polynomials themselves (Duke Mathematical Journal, vol. 2 (1936), pp. 423–434). The present note is based on a corresponding observation with regard to the orthogonal polynomials defined by Szegö for a curve in the plane of a single complex variable. (Received March 4, 1937.)

178. Dr. C. C. Hurd: *Asymptotic theory of linear differential equations containing two parameters.*

This paper considers linear differential equations of order n whose coefficients, regarded as functions of x, are indefinitely differentiable for x on (a, b). These coefficients may be expanded in double series in negative integral powers of a complex parameter λ_1 and a complex parameter λ_2, with possibly a finite number of positive powers of λ_1 and λ_2 present. Under the assumption that the roots of the characteristic equation are simple for every x on (a, b), the existence of a full set of formal solutions is proved. The method of iterations, developed by Trjitzinsky for differential equations, is used to establish the asymptotic character of the solutions. (Received March 4, 1937.)

179. Mr. Olaf Helmer: *Questions of continuity of functions in infinitely many variables.*

A sequence of points $x^{(k)}=(x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)}, \cdots)$, with complex $x_n^{(k)}$, of the space S_∞ is said to *converge weakly* (or *strongly*, or *monotonely*) to $x=(x_1, x_2, \cdots, x_n, \cdots)$ if $x_n^{(k)}\rightarrow x_n$ (or $|x-x^{(k)}|\rightarrow 0$ where $|x|^2=\sum|x_n|^2$, or $x_n^{(k)}\rightarrow x_n$ in such a way that $|x_n^{(k+1)}-x_n|\leqq|x_n^{(k)}-x_n|$, respectively). A function $f(x)$ is said to be *analytic* in the sub-range A of S_∞, if it has the form of a power series in $x_1, x_2, \cdots, x_n, \cdots$ which is absolutely convergent in A; $f(x)$ is called *hyper-continuous* (or *continuous*, or *mono-continuous*) at x, if $f(x^{(k)})<\infty$ for at least one sequence of points $x^{(k)}\neq x$, and if $f(x^{(k)})\rightarrow f(x)$, whenever $f(x^{(k)})<\infty$ and $x^{(k)}$ tends weakly (or strongly, or monotonely, respectively) to x. Hyper-continuity implies both continuity and mono-continuity, but it is seldom found, even among analytic functions. Mono-continuity seems a much stronger requirement than continuity, and yet every analytic fuction is mono-continuous, though not necessarily continuous (for example $\sum x_n$ at $x=0$). On the other hand, there are non-analytic functions which are continuous and not mono-continuous. Example: $F(x)=\sum r^4(x_n)$, where $r(t)=0$ if t is irrational and $1/q$ if $t=p/q$ (in reduced form); at certain points x whose coordinates x_n are algebraic of degree 2, $F(x)$ is continuous (by Liouville's well known theorem on the approximation of irrational numbers), but it is not mono-continuous there. (Received March 6, 1937.)

180. Professor G. Y. Rainich: *Conditional Invariants.*

An expression $f(a)$ is called a conditional invariant under T if the value of $f(a)$ is not affected by the application of T to a provided the a's *satisfy certain relations* R (which must be non-conditionally invariant under T). Although conditional invariants may be reduced to non-conditional ones by elimination, such a reduction may involve, even in rational cases, irrational operations, and a direct treatment seems desirable. When the invariants are polynomials and the transformations form a continuous group it can be proved under certain conditions that given a conditional invariant it can be replaced by a non-conditional one which takes the same values as the given one when the variables satisfy the conditions R. However for fractional invariants cases exist where a non-conditional invariant cannot be replaced in this way. Such invariants are called *singular*. The conceptions introduced here are extended without difficulty to covariants and also to differential invariants and covariants. In translating a special case of Dirac's equations into an equivalent system in vectors Fuller (abstract 42-11-445) was led to certain expressions which can be shown to be singular conditional differential invariants. (Received March 5, 1937.)

181. Mr. C. D. Jones: *On a generalization of Bonnet's theorem.*

For some time it has been known that a sub-variety of m dimension, V_m, in a Riemann spaces of n dimensions, R_n, may be characterized by a set of tensors of even rank. Necessary and sufficient conditions for a set of such tensors to determine a variety have not been stated explicitly in terms of these tensors. This is probably due to the fact that expressions for all of the coefficients in the differential equations which must be integrated to find the V_m have not been found. In this paper are found expressions for all of the coefficients (in particular those which are analogous to the classical three-index symbols) in terms of tensors. As a consequence the integrability conditions of the differential equations may be stated explicitly in terms of the tensors; this leads directly to a generalization of the Bonnett theorem. Tensors which may be considered as a generalization of the Riemann tensor appear in this paper. Their expressions are analogous to those for the Riemann tensor. Complete analysis of the problem leads to two cases wich have been considered separately. An interesting classification of varieties may be made through the various degrees of degeneracy. (Received March 5, 1937.)

182. Dr. D. G. Fulton: *Generalizations of the Cauchy integral formula.*

This paper deals with sets of n functions in n variables which possess the property that they can be evaluated at an inner point of an n-dimensional region T in terms of their values on a hypersurface S which bounds T. For $n=2$ the real and imaginary parts of an analytic function constitute such a set, and the Cauchy integral formula solves the problem of evaluation. In a joint paper with G. Y. Rainich (American Journal of Mathematics, vol. 54 (1932), pp. 235–241) direct generalizations of this formula for higher dimensions are given which solve the above problem for Volterra's conjugate functions. In this paper more general sets of functions are considered which are derived from the earlier ones by the method of transformations, applied both to the functions and the variables. (This method was used by H. C. Chang, Science Society of China, vol. 7, No. 2 (1931)). For these more general functions it is demonstrated that the integrals $\int r^{-n} N^{i\lambda}_{\sigma v} X_\lambda x_\sigma \alpha_v dw$, $r^2 = g_{\alpha\beta} x_\alpha x_\beta$, will give the required evaluations provided the functions X_i are differentiable and satisfy the differential equations $N^{i\lambda}_{jv} \partial X_\lambda / \partial x_v = 0$. The constant coefficients satisfy certain relations. (Received March 5, 1937.)

183. Professor C. G. Latimer: *Quaternion rings.*

A sub-set $\mathfrak{R}$ of a set of integral elements in a generalized quaternion algebra is called a ring if it contains the modulus and is closed under addition, subtraction, and multiplication. The sub-set $\mathfrak{R}$ is said to be primary if no rational prime divides the discriminant of every element in $\mathfrak{R}$. In this paper the basal elements of every primary ring in a given integral set and also basal elements of every integral set containing a given primary ring are determined. From the latter, the number of such integral sets is obtained, a result recently obtained by Eichler by another method (Journal für Mathematik, vol. 174 (1936), p. 159). Every primary ring is uniquely determined, in the sense of equivalence, by two arithmetic invariants. By employing the above results, theorems on the existence of a g.c.d. and on factorization similar to well known results of Hurwitz on the classic quaternion algebra are obtained for a certain type of ring. (Received March 4, 1937.)

184. Professor R. V. Churchill: *The solution of boundary value problems in physics by means of the Laplace transformation.* Part I. *A theory for establishing a solution for problems with vanishing initial conditions.*

The process of solving linear boundary value problems in partial differential equations by first translating them into simpler problems by applying the Laplace transformation has been used since 1923. In problems with vanishing initial conditions the formal steps of this process include the operational method of Heaviside. As in all general methods the proof that the result satisfies all the conditions imposed by the method is usually more difficult than the testing of the result to see if it satisfies all conditions of the problem. A theory is established here for carrying out this process of verification for a broad class of problems, by examining the Laplace transform of the result rather than the result itself. The principal theorems give sufficient conditions on the solution f of the simpler transformed problem under which its inverse transform F satisfies the conditions of the original problem. The solution F has the form of a real infinite integral. Parts II and III of the paper will contain examples of this theory and the theory and examples for problems with non-vanishing initial conditions. (Received March 1, 1937.)

185. Dr. Ralph Hull: *Note on the ideals of cyclic algebras.*

The purpose of this note is the proof of the following theorem. Let $\mathfrak{o}$ be any maximal order of a rational cyclic algebra D of prime degree n, and discriminant $\Delta(D)$. Let $A=A_1A_0$ be any positive integer, where A_0 is prime to $\Delta(D)$ and every prime factor of A_1 divides $\Delta(D)$. Then the number of integral $\mathfrak{o}$-right ideals of norm A is equal to the number of classes of right associated integral matrices of degree n and determinant A_0. If, for example, D is the algebra of ordinary rational quaternions, $\Delta(D)=-4$ and the number given by the theorem is easily seen to be the sum of the positive odd divisors of A. (Received March 2, 1937.)

186. Mr. Louis Green: *Systems of quadrics associated with a point of a surface.*

Relations between an analytic surface immersed in a three-dimensional space and quadrics having second-order contact with the surface are considered in this paper. A new simple characterization of the quadric of Wilczynski is found, as well as a geometric correspondence between the quadrics of Darboux and the transformations of Čech. By a consideration of three special pencils of quadrics related to the quadrics of Darboux a number of canonical lines are characterized. The Moutard pencils of quadrics are also discussed and by means of them certain quadric cones and canonical lines are obtained. Finally, the problem of determining the envelope of a two parameter family of Moutard quadrics, one defined at each point of a surface, is solved. (Received March 2, 1937.)

187. Mr. R. W. Wagner: *Linear matrix functions.*

The matrix functions commonly discussed in the literature are numerical functions with the argument replaced by a matrix. This paper takes the more general point of view that a pointwise correspondence of matrix space with a subset of itself is a matrix function. This discussion is limited to linear functions. The general linear function of the variable matrix X is expressible in the form $\sum_{i=1}^{n^2} A_i X B_i$, where A_i and B_i are constant matrices and the B_i are linearly independent. This expression is not unique; for if the set B_i is replaced by an arbitrary independent set B_i', there is a corresponding set A_i' such that $\sum_{i=1}^{n^2} A_i X B_i \equiv \sum_{i=1}^{n^2} A_i' X B_i'$. The matrix Γ which expresses the linear dependence of the A_i upon the B_i is thus subjected to the transformation $\Gamma' = \alpha^T \Gamma \alpha$. The expression is standardized by giving Γ a canonical form. A change of coordinates of the vector space upon which the matrices operate induces a similarity transformation in the matrix space. In such transformations the A_i and the B_i both transform as X transforms, and Γ is invariant. Certain properties of the function are given by properties of Γ. For instance, the rank of Γ is the number of terms required to express the function. (Received March 5, 1937.)

188. Professor A. H. Copeland: *Postulates for the theory of probability.*

An important question in the theory of probability is the problem of application. The postulate system developed in this paper gives a method of treatment of the problem in that it displays the connections between theoretical probabilities, experimental observations, and predictions. These connections are described in terms of an inner structure of the elements defined by conditional probabilities with respect to a certain subset of the elements. The structure consists in the order of the ω-series and a limit property which corresponds to the limit of the success ratio. The limit postulate restricts the mathematical treatment of the theory of probability without being an assumption as to the nature of the physical universe. (Received March 5, 1937.)

189. Professor R. E. Langer: *On the connection formulas and the solutions of the wave equation.*

A somewhat general discussion is given of the wave equation for a particle with one degree of freedom; that is, (1) $d^2u/dx^2+Q(x)u=0$, with $Q(x)=2m[E-V(x)]\hbar^2$. The asymptotic representability of the solutions $u(x)$ of (1), in terms of single valued functions $U(x)$ which solve an equation closely resembling (1), has previously been shown. It is shown here that coefficients $\alpha_n(x)$ may be successively determined to yield a representation $u(x)=U(x)\sum_{n=0}^{\infty}\alpha_{2n}(x)/\hbar^{2n}+U'(x)\sum_{n=0}^{\infty}\alpha_{2n+1}(x)/\hbar^{2n+1}$, which has advantages over the familiar asymptotic series based upon functions of exponential type. The radial equation for motion in a central Coulomb field is of type (1) with $Q(x)=2m[E+z/x]/\hbar^2-l(l+1)/x^2$. The theory of equation (1) applied to this case has been found consistently to give results at variance with experiment, the "correction" requiring $l(l+1)$ to be replaced by $(l+1/2)^2$. It is shown that this apparent "failure" is due to a misapplication of the theory, and the correct procedure is indicated. Other matters related to the wave equation are also discussed. (Received March 4, 1937.)

190. Dr. E. C. Stopher, Jr.: *Point set operators and their interrelations.*

Several important operators on point sets are defined in terms of an additive derived set operator for which derived sets are closed. Product operators of the form $\phi_1\phi_2\phi_3 \cdots \phi_nA$, where A is an arbitrary set of points and the ϕ's are operators of the family studied, are considered in this paper and many relationships between product operators are established. The order of a product operator is defined and a table is given which includes all product operators of order three or less. Under the additional postulate that the space S is equal to its derived set, it is proved that all product operators of a given family can be given a certain canonical representation. (Received March 4, 1937.)

191. Professor Rufus Oldenburger: *Forms in n variables whose ranks equal n.*

In earlier papers the author proved invariant properties of rank of a form F and its matrix A, where these ranks are defined in terms of generalized determinant minors of any signancy of matrices called "derivates" associated with A. The author also studied an invariant rank not defined in terms of generalized determinants, called "factorization rank." In the case of ordinary matrices these ranks reduce to the rank of the matrix. These are the only arithmetic invariants of a general matrix which have been found to possess this property. In the present paper it is proved that the class C of forms in n variables of any degree p for which the invariant ranks all equal n is the same as the class of forms which are equivalent to $\lambda_i x_i^p$, $i=1, \cdots, n$, $\lambda_i\neq 0$. Since it is at present difficult to determine the factorization rank of a given matrix, a set of properties are obtained which characterize the class C and are readily applied to determine whether or not a particular form is in C. It is proved that if certain simple properties regarding *higher dimensional determinants* are satisfied, a form of odd degree belongs to C. (Received March 5, 1937.)

192. Mr. A. C. Olshen: *Transformations of the Pearson type III distribution.*

Assuming a system of variates $x_1, x_2, \cdots, x_n$ distributed according to the law $y=y_0x^{\alpha\bar{x}-1}a^{-\gamma x}$, $\gamma=2\mu_2/\mu_3$, the author investigates the properties of the distribution of x' obtained by setting $x'=\phi(x)$. Both power and exponential transformations are considered. Properties of the distributions obtained by making power transformations of a set of variates distributed according to the type III law with mean zero and unit variance are also considered. By equating densities, a transformation $x=\Psi(t)$ is obtained, such that for a given frequency distribution $f(x)$, $f(x)=Ca^{-t^2/2}$ is obtained. (Received March 4, 1937.)

193. Professor W. S. Kimball: *Applications of derivatives of line integrals.*

The new formulas for derivatives of line integrals recently reported are applied to various integrals. When the integral is independent of the path, the first and higher derivatives are identically zero. When the path is an extremal arc, the first derivatives are zero as well as the area derivative. The second partials with respect to x have the same sign as those with respect to y, being negative for a maximum and positive for a minimum. The order of differentiation is indifferent if and only if the path of integration is an extremal arc, and the sign of this second partial with respect to both x and y is equal or opposite to the sign of the second partials with respect to x and y according as y' is negative or positive. Line integrals with no y' present in the vector integrand have ordinarily a characteristic pattern of null lines (along which the integral is zero) and extremal arcs which is independent of the integration limits. Where y' is present in the vector integrand as with the calculus of variations, there is ordinarily a two parameter family of such characteristic patterns dependent on the two integration limits. (Received March 3, 1937.)

194. Professor L. M. Blumenthal: *Note concerning Jordan metrics.*

In a short note (Casopis Matematiky a Fysiky, vol. 64 (1934–35), pp. 182–183), D. Barbilian has introduced the following interesting metric: Let A, B be any two points of the interior J of the simple closed plane curve K, and attach to this pair of points the number log (M/N), where M and N denote the maximum and minimum values, respectively, of the function PA/PB, P an element of K, and PX the euclidean distance of the points P, X. Barbilian asserts that the point set J, so metrized, is a metric space, and theorems are announced, without proof, concerning the geometry of this space. In the present note, the proofs of these theorems are furnished, together with some additional properties of the space. It was found useful to consider J a subset of the Gauss plane, the metric then being $(AB)=\max_{z',z''\epsilon K} \log \left|(x_1, x_2, z', z'')\right|$, where x_1 and x_2 are the complex numbers attached to A and B respectively, and (x_1, x_2, z', z'') indicates the cross ratio of the four complex numbers. (Received March 5, 1937.)

195. Mr. H. A. Luther: *A third order irregular boundary value problem.*

The problem treated is that of expanding functions in infinite series whose terms are solutions of the differential equation $d^3u/dx^3+\rho^3u=0$ and certain highly irregular boundary conditions. The distinctive feature of the boundary conditions is that they are linear homogeneous forms in the derivatives of u evaluated at the end points of an interval (a, b), and that at least two of the three boundary conditions bear at both a and b. It is found that analyticity of a given function insures uniform convergence of its expansion in a region of restricted size centered at $(b+2a)/3$. However, if one seeks convergence in a larger region, the given function must obey conditions dependent on the specific character of the boundary conditions. (Received March 5, 1937.)

196. Professor A. A. Albert: *Symmetric matrices in a modular field.*

A symmetric matrix with elements in a field F of characteristic two is called *null-symmetric* or *diagonal-symmetric* according as its diagonal elements are or are not all zero. Null-symmetric matrices are shown to be congruent in F if and only if they have the same rank, and are never congruent to diagonal-symmetric matrices. The latter type is always reducible to diagonal form as in the classical theory. The results are applied to quadratic forms and to orthogonal equivalence. It is shown that two symmetric matrices with elements in an algebraically closed field of characteristic not two are orthogonally equivalent if and only if they are similar. This result is false for the characteristic two case, but an orthogonal reduction theory in an arbitrary field F of characteristic two is obtained, and the true reason for the falsity given. Finally the nature of the invariant factors of any symmetric matrix in F of characteristic two is completely determined. (Received March 1, 1937.)

197. Professor H. P. Thielman: *A generalization of trigonometry.*

The underlying hypothesis of this note is the existence of a class of single valued functions $F(x)$ and an operation "$\cdot$", called multiplication, which satisfies the following laws: it is associative, commutative, distributive with respect to ordinary addition, and is such that if k is independent of x then $F(x)\cdot kG(x) =kF(x)\cdot G(x)=F(x)\cdot G(x)k$. One of the functions of the given class, say $f(x)$, is assumed to have the properties $f(x)\cdot f(y)=f(x+y)$, $f(0)\neq 0$. Next two functions are defined: $s(x)=[f(x)-f(-x)]/2k$, $c(x)=[f(x)+f(-x)]/2$. On this basis the addition formulas, a generalized DeMoivre's theorem, and various other formulas and theorems are obtained for $s(x)$ and $c(x)$. This theory reduces to the ordinary circular trigonometry if "$\cdot$" stands for ordinary multiplication, $f(x)=e^{ix}$, and $k=i$. For other values of $f(x)$ and k elliptical and hyperbolic trigonometries are obtained. If "$\cdot$" stands for Volterra's composition of functions of the "closed cycle," $f(a)=e^{ial^*}$, $k=1$, a trigonometry is obtained which includes the theory of the Bessel functions ber x, and bei x. (For a definition of e^{ial^*} see V. Volterra, *Theory of functionals*, translated by Miss M. Long, p. 104.) If "$\cdot$" stands for the composition of functions in n variables, a trigonometry of hypergeometric series results. (For the definition of composition of functions in n variables see H. P. Thielman, Annals of Mathematics, vol. 31 (1930), p. 198.) (Received March 5, 1937.)

198. Dr. A. E. Ross: *On certain rational transformations.*

Canonical quadratic forms under rational transformations are well known. Since in the case of homogeneous diophantine equations existence of a rational solution implies existence of an integral solution, these canonical forms may be used to obtain criteria of solvability in integers of such equations (A. Meyer, Journal für Mathematik, vol. 98 (1885), p. 177, and Dickson, *Studies in the theory of numbers*, p. 70). It is shown that in the case of non-homogeneous diophantine equations of degree two it is possible to choose parameters in the above mentioned rational transformations to afford a similar reduction of a general problem to the simpler case of either a canonical form or an incomplete canonical form. (Received March 5, 1937.)

199. Dr. J. L. Brenner: *The derivative of a rational function.*

Let f and g be two binary forms and let a and b be distinct zeros of f. If C_i traces out the distinct roots of fg with the exception of a and b, and if r_i traces out the zeros of the Jacobian of f and g which are distinct from a, b, and $\{c_i\}$, then the product of the cross-ratios $\prod_i(ab; c_ir_i) = -1$. From this theorem many new and old facts about the derivative of a rational function can be proved. For example, form the rational function f/g where f and g are arbitrary except that their degrees, $M(M>2)$ and N, are fixed and the zeros of f/g all lie in a circle of radius R. At least one pole of f/g or a zero of $d(f/g)/dz$ will lie in a concentric circle of radius S, where S/R depends only on $M+N$. If f and g are of arbitrary degree and have real coefficients, then the number of distinct real zeros of $d(f/g)/dz$ is at least as great as the absolute value of the difference between the number of distinct real zeros and the number of distinct real poles of f/g. (Received February 16, 1937.)

200. Professor Otto Szász: *On the partial sums of Fourier series.*

Very short proofs are given for two theorems of Paley, with generalizations and improvements upon the contents. The main results are: (1) If $\phi(x)\sim a_0/2+\sum_1^\infty a_v \cos vx$, $a_v \geqq 0$, $v=1, 2, 3, \cdots$, and if $\liminf_{x\to 0} 2\left[\int_0^x \phi(t)(x-t)dt\right]/x^2 \leqq M$, then $\sum a_v < \infty$, and $a_0/2+\sum_1^\infty a_v \leqq M$. (2) If $\omega(x)\sim\sum_1^\infty b_v \sin vx$, $b_v \geqq 0$, $v=1, 2, 3, \cdots$, and if $\left[\int_0^x \omega(t)dt\right]/x \leqq M$ for $0<x\leqq 2\alpha_0$, then $\sum_0^n v b_v \leqq \alpha_0 Mn/\sin^2\alpha_0$, $n=1, 2, 3, \cdots$, where $\alpha_0=1.16\cdots$ is the unique root of the equation $2\alpha=\tan\alpha$ in $0<\alpha<\pi/2$; and $\alpha_0\sin^2\alpha_0<1.38$. If in addition $|\omega(x)|\leqq M$, $0<x<\pi$, then $\left|\sum_1^n b_v \sin vx\right| \leqq M(1+\alpha_0/\sin^2\alpha_0)<2.38M$, $n=1, 2, 3, \cdots$. The method can be applied to other generalizations of Paley's theorems, and also to almost periodic functions. (Received February 16, 1937.)

201. Dr. G. M. Ewing: *Sufficient conditions for a discontinuous solution in the calculus of variations.*

Let E be an extremaloid with one corner for $J=\int f(x, y, y')dx$. It is shown that E furnishes a proper strong relative minimum if each arc of E of class C'

satisfies the conditions I, II_b', III′, and IV′, and if the Erdmann corner conditions (V) are satisfied. The statement of II_b' is modified so as to permit the vanishing of the E-function along a certain curve Γ through the corner, but I, III′, and IV′ are taken in the usual form. A condition (VI) on the Carathéodory corner function ω is introduced, which is much less restrictive than the assumption that $\omega \neq 0$ at the corner and it is found that I, II_b, III′, IV′, V, and VI are sufficient for an improper minimum. The method used also applies to an extremaloid with n corners. (Received February 22, 1937.)

202. Dr. S. B. Myers: *Rectifiability of arcs in Finsler space.*

As is well known, in euclidean space the length of an arc $x^i = x^i(t)$ as defined by $\int \sqrt{\sum (dx^i/dt)^2} dt$ is equal to $\lim \sum d(A_j A_{j+1})$, where A_k is a point on the arc, d represents distance, and the limit is taken as the subdivisions of the arc by the set (A) become finer. This is also true if $x^i(t)$ have bounded, Riemann integrable derivatives. The object of this paper is to prove these and similar theorems for regular Finsler spaces. These are n-dimensional manifolds of class C^3 with arc length defined by an invariant integral $\int F(x^1, \cdots, x^n, x^{1\prime}, \cdots, x^{n\prime}) dt$, where F is positive, of class C^2, positive homogeneous of first degree, and $|F_{x^{i\prime} x^{j\prime}}|$ is of rank $n-1$, while distance is defined as the greatest lower bound of lengths of arcs joining two points. (Received March 3, 1937.)

203. Professor E. W. Miller: *Concerning biconnected sets.*

In their paper on connected sets (Fundamenta Mathematicae, vol. 2 (1921), pp. 206–255) B. Knaster and C. Kuratowski introduced the idea of a biconnected set and gave several examples of such sets. Each of the biconnected sets which they constructed contains a dispersion point and Kuratowski raised the question whether every biconnected set contains such a point (Fundamenta Mathematicae, vol. 3 (1922), p. 322). The main object of the present paper is to prove that if the hypothesis of the continuum is true, there exists (in a bounded portion of the euclidean plane) a biconnected set which contains no dispersion point. The proof is based on a procedure for the construction of widely connected sets due to P. M. Swingle (this Bulletin, vol. 37 (1931), pp. 264–268) and on a property of families of sets which has been called by the present writer property B (see abstract 42-5-199). Extensive use is made of the axiom of Zermelo. (Received March 3, 1937.)

204. Professor E. W. Miller: *On a theorem due to A. Mullikin.*

In this paper it is proved that if C is a compact metric continuum and F_1 and F_2 are non-vacuous mutually exclusive closed subsets of C, then there is a *constituent* of $C-(F_1+F_2)$ which has a limit point in F_1 and a limit point in F_2. This extends a well known result due to Mullikin (Transactions of this Society, vol. 24 (1922), pp. 144–162). Mullikin's theorem in this stronger form seems well adapted to the investigation of certain general problems concerning collections of continua. In this paper the theorem is used to extend Sierpinski's well known result concerning the sum of a countable infinity of disjoint closed sets (Tôhoku Mathematical Journal, vol. 13 (1918), pp. 300–303). Results relating to the common boundary of two plane domains are also obtained. (Received March 3, 1937.)

205. Dr. E. C. Stopher, Jr.: *Cyclic relations in point set theory.*

This paper is concerned with the formula $\phi c\phi c\phi c\phi A = \phi c\phi A$, where A is an arbitrary point set in a space S, ϕ is an arbitrary point set operator, and c denotes the operation of taking complements. The formula holds for closure (Kuratowski), interior (Zarycki), and derived set (S. T. Sanders, Jr.) operators. The author establishes the formulas for a number of additional important point set operators defined in terms of a general postulated derived set operator. (Received March 4, 1937.)

206. Professor V. G. Grove: *Geometry of a surface in the neighborhood of a spine.*

A point P on a surface S will be called a spine if there exists a unique line t through the point such that every section of the surface through t has an ordinary cusp at P with t as cuspidal tangent. The line t is called the spinal tangent of S at P. The author studies S by considering sections of the surface through t. The results are obtained by reducing each of the equations of the surface S and the equations of a curve with a cusp to canonical forms. Complete geometrical characterizations are given for the reference systems giving rise to each of these cannonical forms. (Received March 4, 1937.)

207. Dr. Max Herzberger: *The characteristic function in the history of optical problems.*

A survey is given of the development of the idea of the characteristic function and its importance for optical problems. The method and principal ideas of Hamilton are sketched. The changes are noted which occurred at the rediscovery in 1895 by Bruns. The advantages and disadvantages of these changes are discussed. The so-called Seidel eiconal of Schwarzschild is described, the introduction of new variables by T. T. Smith, and finally the introduction of so-called natural variables by the author are sketched. The author will try to outline the applicability of the methods found in optics for other variation principles. (Received March 5, 1937.)

208. Dr. C. B. Allendoerfer: *The algebraic determination of the class of a Riemann space.*

The possibility of imbedding an n-dimensional Riemann space in an $(n+p)$-dimensional Euclidean space has been previously stated in terms of the simultaneous solution of a system of partial differential equations known as the Codazzi equations and a system of algebraic equations known as the Gauss equations. The author has recently defined the type τ_p of such a space and has shown that if $\tau_p \geqq 3$ the solution of the Gauss equations, if any, is unique. In this paper the actual solution of these equations is obtained when $\tau_p \geqq 3$, and in this case there is derived a set of algebraic conditions on the g_{ik} of the space and their derivatives which are necessary and sufficient that the imbedding be possible. It is also shown that these conditions are necessary and sufficient that the space actually be of class p. (Received March 5, 1937.)

209. Mr. J. W. Calkin: *Some self-adjoint boundary value problems.*

The notation of a previous note (abstract 43-3-114) is preserved. The results announced there yield obvious theorems on the solution of the systems $Tf=g$, $f(s)=0$, and $Tf=g$, $p\partial f/\partial n+Af(s)=0$, where A is a bounded self-adjoint operator in $\mathfrak{L}_2(C)$. It is a familiar fact that the solution of the systems (1) $Tf=0, f(s)=h(s)$, and (2) $Tf=0$, $p\partial f/\partial n+Af(s)=h(s)$, is a closely related problem. For $q\geqq 0$ on E, the following results are obtained: (1) has a unique solution f for every h in $\mathfrak{L}_2(C)$. For $h(s)=0$, (2) has at most a finite number of linearly independent solutions f in $\mathfrak{D}^*$. If $\mathfrak{M}$ is the manifold in $\mathfrak{L}_2(C)$ whose elements are the boundary values $f(s)$ of such solutions, (2) has a solution f in $\mathfrak{D}^*$ for every h in $\mathfrak{L}_2(C)\ominus\mathfrak{M}$, which is uniquely determined by h and the condition $f(s)\perp\mathfrak{M}$. These results do not apply to the Dirichlet and Neumann problems of potential theory. The known facts have been used there to obtain necessary information about the domain $\mathfrak{D}^*$. The precise sense in which $f(s)$ and the derivative $\partial f/\partial n$ exist will be described in a more detailed report on this work. (Received March 5, 1937.)

210. Miss R. G. Simond: *Relations between certain continuous transformations of sets.*

True arc-preserving, tree-preserving, true tree-preserving, strongly monotonic, strongly irreducible, and contracting transformations are defined in this paper and are studied in connection with arc-preserving, irreducible, monotonic, and homeomorphic transformations. Given $T(A)=B$, where T is a single valued continuous transformation, different sets of conditions have been found which are sufficient to make T a homeomorphism on A. It is shown that if A is a locally connected continuum and if T is true arc-preserving and strongly irreducible on A, then T is a homeomorphism on A. New conditions for T to be a homeomorphism on A, where A is a cyclicly connected set, have also been found. Other relations between these transformations include the following results. If A is a locally connected continuum and T is arc-preserving on A, then T is tree-preserving on A. If A is a tree and T is true arc-preserving on A, then T is contracting on A. (Received March 5, 1937.)

211. Dr. H. E. Vaughan: *On the problem of imbedding two abstract spaces in a third.*

The type of the problems treated in this paper is the following: Let (P_1, K_1) and (P_2, K_2) be abstract spaces (P_1 and P_2 being sets, K_1 and K_2 operations of derivation) such that P_1 and P_2 have elements in common. Determine under what conditions there will exist a space (P, K), where $P=P_1+P_2$, such that the identical mappings of (P_1, K_1) and (P_2, K_2) in (P, K) are homeomorphisms. When such spaces exist, define the operation K in terms of K_1 and K_2 in the "weakest" manner consistent with the above requirement. Determine the restrictions which must be placed on the intersection of (P_1, K_1) and (P_2, K_2) in order that properties of K_1 and K_2 (such as distributivity or closedness) will also be possessed by the weak sum so defined. Repeat the above considerations with the restriction that all spaces involved belong to a smaller class (for example, that of accessible or H-spaces). (Received March 5, 1937.)

212. Dr. M. E. Shanks: *On conformal mapping.*

If J is a rectifiable Jordan curve it is known that a conformal transformation of the interior of J on the interior of the unit circle C carries sets of measure zero on J into sets of measure zero on C and conversely. This result is reestablished using Carathéodory linear measure, and is extended to Jordan curves whose points of non-rectifiability are at most countable. A one-to-one continuous mapping of C on any Jordan curve J determines, under suitable restrictions, a conformal mapping of the interiors. This same boundary mapping determines a conformal transformation of the exterior of C on the interior of $\bar{J}$, where $\bar{J}$ is the conjugate of J. (Received March 5, 1937.)

213. Dr. W. T. Reid: *A direct expansion proof of sufficient conditions for the non-parametric problem of Bolza.*

In a recent paper (abstract 41-11-379; to appear in the Annals of Mathematics) the author has proved by expansion methods a sufficiency theorem for the general non-parametric problem of Bolza that had previously been established by Hestenes (Transactions of this Society, vol. 36 (1934), pp. 793–819) using the classical field method. This proof had in common with the previous field method proof the property that the theorem was not established directly for the general problem with non-separated end-conditions. Instead, such a problem was first transformed into an equivalent one in a greater number of dependent variables and with separated end-conditions. The present note gives a direct expansion proof of sufficient conditions for the general non-parametric problem of Bolza. This direct expansion proof employs a certain extension of the transformation of Clebsch, which is essentially the same as an auxiliary theorem used by Hestenes (abstract 42-11-450) in a direct proof by field methods of a sufficiency theorem for the general problem of Bolza. (Received March 5, 1937.)

214. Reverend W. C. Doyle: *On Garvin's $R(n)$ function.*

In a paper by Garvin (American Journal of Mathematics, vol. 58 (1936), p. 510) is found a generalized Möbius function $R(n)$ defined in such a way that the preceding $R(n)$'s must be known in order to find any particular $R(n)$. Here is given a formula for $R(n)$ which is such that the preceding ones of the set need not be known. This relation may also be taken as a new definition of $R(n)$. (Received March 5, 1937.)

215. Dr. R. S. Martin: *Note on harmonic functions represented by a Daniell integral.*

N. Wiener has shown that an extensive class of functions harmonic within a region T for which the continuous Dirichlet problem is solvable can be represented by a Daniell integral $I_p(f)$ where f belongs to a class of summable functions defined on the boundary t of T. Wiener proves that if f is summable, non-negative, and *bounded*, and if it vanishes throughout a relative neighborhood of a point Q of t, then $I_p(f)$ approaches zero as P approaches Q. He remarks, however, that the hypothesis of boundedness is possibly superfluous. An example is constructed of a three dimensional region T which admits solution of the continuous Dirichlet problem, and which, incidentally, has for a boundary a simple closed surface of finite area. For this region there is an (unbounded) non-negative function f_0 defined on the boundary t of T and vanishing throughout a relative neighborhood of a point Q of t. The corresponding $I_p(f_0)$ exists and has an infinite superior limit as P approaches Q. (Received March 6, 1937.)

216. Dr. P. O. Bell: *Geometric characterizations in projective differential geometry of curved surfaces.*

At a general point y of a curved surface S_y two ruled surfaces may be generated by the asymptotic tangents at y to the surface as the point y moves in an arbitrary direction. A study of these ruled surfaces in relation to the surface S_y for variously defined directions yields a number of interesting results. New characterizations of the curves of Darboux and Segre are obtained and a number of theorems concerning them are given. New characterizations are also given for the projective normal, the directrices of Wilczynski, and adjoint union curves. (Received March 6, 1937.)

217. Professor L. W. Cohen: *Uniform continuity in non-metric spaces.*

In this note the notion of uniform continuity is generalized to apply to Hausdorf space satisfying the first denumerability axiom. The theorem concerning the extension of a function uniformly continuous on a set to a function continuous on the closure of the set is proved. The spaces of the domain of definition and of the value domain of the function need not be metrizable. The result is a continuation of the preliminary report on Cauchy convergence in topological space (abstract 41-11-399). (Received March 6, 1937.)

218. Dr. O. G. Harrold, Jr.: *Concerning the convexification of continuous curves.* I.

In this paper extensions are made to the previously known results concerning curves which can be convexified in the sense of Menger (*Untersuchungen über allgemeine Metrik*, Mathematische Annalen, vol. 100 (1928)) to the following types of compact, locally connected continua: (a) curves with a finite number of im kleinen cycle points; (b) curves with a finite number of complementary domains; (c) "beständig regular" curves. (Received February 27, 1937.)

219. Professor B. A. Bernstein: *Remarks on Nicod's reduction of Principia Mathematica.*

The author points out certain obscurities in Nicod's definitions, in terms of "$|$", of the *Principia's* $\sim p$, $p \supset q$, $p \vee q$, $p \cdot q$. In consequence of these obscurities, Nicod's proofs of the Principia propositions are incomplete. The author clears up the obscurities, and completes the Nicod proofs. (Received March 4, 1937.)

220. Mr. N. A. Hall: *On binary quadratic forms with a single class in each genus.*

The form of the number of representations function for positive definite binary quadratic forms suggests a consideration of the number of cases in which there is a single class in each genus. It is found that whenever the discriminant satisfies certain significant congruences, and whenever it contains prime square factors >2, there is a finite number of such cases. These results simplify materially the problem of obtaining a general expression for the number of representations function. (Received March 4, 1937.)

221. Dr. A. E. Taylor: *The abstract theory of analytic functions, and certain special cases.*

A function $u(x, y)$ on $[E(R)]^2$ to $E'(R)$ (Banach spaces) is said to be bi-harmonic if it is continuous, with continuous first and second Fréchet differentials such that $d^2_{xx}u(x, y; \xi; \eta)+d^2_{yy}u(x, y; \xi; \eta)=0$. If u is bi-harmonic in a linearly simply connected domain there exists a second function $v(x, y)$, also bi-harmonic in the domain, such that $u+iv$ is analytic (see abstract 42-9-369.) The set of points in whose neighborhood a convergent sequence of analytic functions (on one Banach space to another) fails to converge uniformly is closed and nowhere dense with respect to the domain in question. If $f(z, \alpha)$ is a complex valued functional of the Banach variable z and the complex number α, analytic in each separately in a "cylinder region" of the product space, then it is analytic also in the pair. If $f(z)$ is a continuous function on a Banach space to the Hilbert space H_0, necessary and sufficient conditions that it be analytic are that the function $W(\zeta, z)=\sum_1^\infty \zeta^{n-1}f_n(z)$ be analytic as a function of z, for each $|\zeta|<1$, where the $f_n(z)$ are the components of $f(z)$ in H_0. (Received March 4, 1937.)

222. Professor E. T. Bell: *Reducible ternary arithmetical cubics.*

All such cubics such that any rational integer has at most a finite number of representations in the form are determined. (Received March 5, 1937.)

223. Miss Dorothy Manning: *On simply transitive groups with transitive Abelian subgroups of the same degree.* II.

If a simply transitive permutation group G of degree p^{a+b+c} (p a prime) contains a transitive Abelian subgroup of the same degree and of type (a, b, c), then G is imprimitive and compound when $a+b<c$. Such groups need not be imprimitive when $a+b=c$. (Received March 5, 1937.)

224. Professor Morgan Ward: *Arithmetical properties of sequences in rings.*

Let (u): $u_0, u_1, \cdots, u_n, \cdots$ be a sequence of elements of a commutative ring O. A study is made of the periodicity and divisibility of such sequences relative to ideals of O, with special emphasis on linear sequences and divisibility sequences. Any (u) sequence is an instance of a correspondence between elements of a commutative ring and of a lattice previously investigated (see the author's abstract 42-9-375; the paper will appear in the Annals of Mathematics). The less general hypotheses of the present paper allow many of the axioms of the earlier paper to be discarded. (Received March 5, 1937.)

225. Mr. H. A. Arnold: *A set of independent postulates for a group with equality undefined.*

A set of independent postulates is given involving equality as an undefined notion. The equality need not be taken as identity. The properties of symmetry are postulated for this generalized equality, together with the following: if $a=b$, then $ca=cb$ and $ac=bc$. Reflexivity is later proved. The existence of at least one solution to the equation $ax=b$ is postulated. It is proved that all others bear the relation of undefined equality to one another. Properties precisely similar to those of the usual group are derived. (Received March 6, 1937.)

226. Mr. Victor Elconin: *Quotient differentials.*

Let $f(x)$ be in a group Γ with respect to xy, x^{-1}, u. The first quotients Q_1, Q_2, Q_3, Q_4 of $f(x)$ are $f(xy)f(x)^{-1}$, $f(x)^{-1}f(xy)$, $f(yx)f(x)^{-1}$, $f(x)^{-1}f(xy)$, in general distinct if Γ is non-abelian. If $\|x\|$ exists, >1 or $=1$ according as $x\neq u$, $x=u$, and x^a exists in Γ such that for all real a, b, $(x^a)^b=x^{ab}$, $x^{a+b}=x^ax^b$, then *quotient differentials* $q_i^{xy}f(x)$ are defined as follows: for any $\epsilon>0$, $\delta>1$ exists such that $\|Q_i^{-1}q_i^{xy}f(x)\|\leqq\|y\|^\epsilon$ if $\|y\|<\delta$, and $q_i^{xy^a}f(x)=(q_i^{xy}f(x))^a$. The q_if are unique if Γ also satisfies the requirement: for some $a, b>0$, $\|x^{-1}y^{-1}\|\leqq\|xy\|^a\leqq(\|x\|\,\|y\|)^b$ for all x, y, and for any x, y, $xy\neq u$, there exist $a, b>0$ such that $a\leqq\|x^cy^c\|^{1/c}\leqq b$ for all $c>0$; then Γ is an *analytical group* with respect to xy, x^a, $\|x\|$. Abelian and non-abelian instances are given. If Γ is analytical, all the q_if are expressible in terms of any one. Applications of the q_if algorithm are given, including several to R. E. Moritz' quotientiation theory. (Received March 6, 1937.)

227. Dr. A. L. Foster: *Arithmetic prime-decomposition and Boolean algebra.*

One difference between the concept prime-element as used in arithmetic $(N,\cdot)$ (and also in many rings), and the concept prime- (or atomic-) element of Boolean algebra is that in the latter case the prime- (or in general the prime-ideal) divisors of an element uniquely determine the element, while in the former case the prime divisors together with their respective multiplicities are required. In a way analogous to the general method of realizing an abstract Boolean algebra as an algebra of classes, in which the concept Boolean-ideal plays the central role (Stone, Transactions of this Society, vol. 40) this paper introduces a simple concept, integer-ideal (of a Boolean algebra), a notion which is appropriate for class-realizing, for instance, $(N,\ \cdot)$. (Received March 6, 1937.)

228. Mr. D. H. Hyers: *A note on linear topological spaces.*

Two apparently different definitions of bounded sets in linear topological spaces L have been given by A. Kolmogoroff (Studia Mathematica, vol. 5 (1934), p. 30) and by J. von Neumann (Transactions of this Society, vol. 37 (1935), p. 7). The author proves that the two definitions are equivalent. Any compact set of L is shown to be bounded. A theorem on the normability of a linear topological space due to Kolmogoroff (loc. cit.) is applied to prove that for L to be finite dimensional it is necessary and sufficient that there exist an open set of L which is convex and compact. The relation between complete linear topological spaces and the spaces of type (F) (for definition see Banach, *Operations Linéaires*, p. 35) is now investigated. By using an important result of G. Birkhoff on the metrizability of topological groups (Compositio Mathematica, vol. 3 (1936), p. 428) it is shown that L may be considered as a space of type (F) if, and only if, L satisfies Hausdorff's first countability axiom. (Received March 6, 1937.)

229. Mr. Francis Dresch: *The general equilibrium and a simplified economic system.*

This paper is concerned with determining the distribution of real income among the factors of production in a total economic system. Strict competition is assumed and certain other assumptions are made as to the technical nature of the process of production. It is shown how, by the construction of satisfactory index numbers, the discussion of such a system with a large number of commodities may be reduced, without substantial loss of generality, to a discussion of a simplified system involving only three distinct commodities and a more convenient number of variables. (Received March 8, 1937.)

230. Professor R. D. James: *A problem in additive number theory.*

In this paper the following results are proved. Consider the set of positive integers n_i such that $n_0=1$ and such that every prime factor of n_i, $i\geqq 1$, is congruent to 1 (mod 4). If $r=-1$, 0, 1, or 2 and if t is any integer $\geqq 1$, then every integer $n\equiv r$ (mod 4) is a sum of *exactly* $r+4t$ integers n_i all but three of which may be taken equal to 1. If $r=-1, 0$, or 1 this result is the best possible in the sense that there are an infinite number of integers which are not the sum of fewer than $r+4$ integers n_i. If $r=2$ it is probable that every integer $n\equiv 2$ (mod 4) is the sum of *exactly* two integers n_i. At present the most that can be said is that every large integer $n\equiv 2$ (mod 4) is the sum of two integers which have all except possibly two of their prime factors congruent to 1 (mod 4). (Received March 6, 1937.)

231. Professor A. D. Michal: *Differential geometry of an abstract topological group with Banach coordinates.* II.

The author continues his studies in the differential geometry of abstract topological groups (see abstract 42-11-470). Special attention is given to groups of motion in the topological groups. The differential geometry of topological semi-groups and other related topological manifolds is also considered. (Received March 6, 1937.)

232. Mr. E. W. Paxson: *A further existence theorem for the ordinary differential equation in a linear topological space.*

The differential equation $dy/d\mu=f(\mu, y)$ (see abstracts 42-9-355 and 42-11-457) is considered from another point of view. The function $f(\mu, y)$ is required to be continuously on the real numbers and the space L to a bounded convex set in L. An abstract function space of continuous functions to L is constructed and a suitable topology introduced for it under which it is of linear type L. Functional continuity of the integral in the integrand having been demonstrated, Tychonoff's extension of the Brouwer fixed point theorem is applied (with $f(\mu, y)$ to a bicompact set) to the integral transformations arising from the differential system, yielding a non-constructive existence theorem. (Received March 6, 1937.)

233. Professor H. F. Blichfeldt: *On diophantine approximations.*

The author considers a set of $n-1$ irrational numbers $\alpha_1, \cdots, \alpha_{n-1}$ and a function $F(\alpha, x, z) \equiv Xf(z/X$, where $X = |\alpha z - x|$, satisfying certain conditions. The problem of finding a superior limit to the $n-1$ numbers $F(\alpha_i, x_i, z)$, under the condition that z and the x_i are integers, is attacked by the method explained in the paper, *A New Principle in the Geometry of Numbers*, Transactions of this Society, 1914. Applied to the problem of approximating the irrationals α_i to fractions x_i/z, this gives for the $|\alpha_i - x_i/z|$ the superior limit $\gamma/z^{n/n-1}$, where for large n the coefficient γ approaches $\pi^{-1/n}$ = approximately $1-1.145/n$. Minkowski's value was $1-1/n$. (Received March 6, 1937.)

234. Dr. M. R. Hestenes: *Quadratic functionals in the calculus of variations.*

In this paper a systematic study is made of quadratic functionals of the type which appear as the second variation of problems in the calculus of variations. Particular attention is given to the study of the second variation for the problem of Bolza. The method used is essentially that of Birkhoff and Hestenes (Duke Mathematical Journal, vol. 1 (1935), pp. 198–286) but with many modifications and simplifications. The results obtained are analogous to those obtained for the simpler problems treated in the earlier paper. In the application of the theory to the problem of Bolza no normality assumptions are used. (Received March 8, 1937.)

235. Mr. R. W. Shephard: *Problems of the incidence of taxation in a simplified economic system.*

As illustrative of the general method, a particular tax of ξ dollars per dollar value on capital used in the production of both consumption and capital goods is considered. The income receivers within the economic system are classified into producers of capital goods, laborers, and producers of consumption goods, with distributive coefficients a, b, and c, respectively, where $a+b+c=1$. Strict competition and a steady state with regard to money is assumed. For the determination of the incidence of this tax, the changes δa, δb, and δc must be found. These may be expressed in terms of the changes in the quantities of labor and capital used in production. A variety of results may be obtained for δa, δb, and δc, depending on certain elasticities describing the production functions. A complete classification of all such possibilities is given by means of these elasticities. (Received March 6, 1937.)

236. Mr. C. D. Olds: *The remainder in the approximate evaluation of the probability in the symmetrical case of James Bernoulli's theorem.*

This paper considers James Bernoulli's theorem concerning the probability P that the number of successes, m, of an event in a series of n independent trials, each with constant probability $p=q=1/2$, should lie between two given limits. The exact expression for P is given in integral form. This integral is then approximated by methods similar to those used by Laplace. An approximation to P is thus obtained together with the upper limit of the error term, or remainder, Δ, and it is shown that $|\Delta| < n^{-2}/2$ for $n \geqq 17$. (Received March 8, 1937.)

237. Professor Glenn James: *Properties of the quotients of the parameters in Fermat's equation.*

This paper studies the ratios of the integers x, y, z, in the Fermat equation, $x^n+y^n=z^n$, under the assumption that they satisfy the equation. In particular it is proved that, as $n\rightarrow\infty$, $\lim y/z=\lim (y/z)^n=\lim [(z-x)/z]^{1/n}=1$, and $\lim x/y=\lim (x/z)^n=\lim [(z-y)/z]^{1/n}=0$. (Received March 8, 1937.)

238. Dr. Alvin Sugar: *Waring's problem for a set of non-negative integers related to the polynomial $m(x^3-x)/6+x$.*

First we show that Dickson's ascension theorem may be generalized in such fashion that it is applicable to any set of non-negative integers. We then apply this ascension theorem to the set of non-negative integers $K_{mt}=[0, 1, \cdots, t-1, m(x^3-x)/6+x+t]$ and prove that every positive integer is a sum of $[(m+t)/(t+1)]+3$ numbers of K_{mt}. This result is a generalization of two results previously obtained by the author ($t=0$, see American Journal of Mathematics, vol. 58 (1936), pp. 783–790; $t=1$, see American Journal of Mathematics, vol. 59 (1937), pp. 43–49). (Received March 8, 1937.)

239. Dr. O. G. Harrold, Jr.: *On the expansion of the remainder in the open-type Newton-Cotes quadrature formula.*

The problem of obtaining an asymptotic expansion for the remainder term to the open-type Newton-Cotes quadrature formula is considered. An expansion is obtained which is similar to the classical development given by the Euler-Maclaurin summation formula. This expansion contains the result obtained by J. F. Steffensen for the remainder term as a special case. The following properties of the Cotes coefficients (A_i) are found to be important in this development: (a) $A_i>0$, $i=2l+1$, $i\leqq n/2$; $A_i<0$, $i=2l$, $i\leqq n/2$; (b) if $n\equiv 3$ (mod 4), $\sum_{i/n<1/2} A_i i/n>1/8$. (Received December 2, 1936.)

240. Mr. N. A. Hall: *Polynomial and semi-polynomial forms of the confluent hypergeometric functions of two variables.*

As a generalization of the results of P. Humbert in connection with the Laguerre polynomials of two variables, the six types of confluent hypergeometric functions of two variables are investigated in the cases where they are polynomials in only one or in both variables, that is, in the semi-polynomial or polynomial forms. These give rise to a great variety of generating functions involving exponential, Bessel, and ordinary confluent hypergeometric functions analogous to those known for the Laguerre and Jacobi polynomials, and also to certain characteristic addition formulas interrelating the several forms. (Received March 4, 1937.)

241. Professor Clifford Bell: *Commutative one-dimensional linear transformations.*

The product transformations T_1T_2, T_2T_1 are formed from the one-dimensional linear transformations T_1, T_2. The condition that the commutative law holds for these products, T_1 being considered fixed, leads to a quartic equation in the constant of proportionality between the coefficients of T_1T_2 and T_2T_1. The roots of the quartic give rise to two general types of transformations T_2. Special cases are considered including those for which T_1, T_2 and the product T_1T_2 are involutions. (Received March 4, 1937.)

242. Professor E. T. Bell: *Representations in certain pure forms of degrees higher than the second.*

The forms in question have numerical coefficients. The integers represented in them are given, at least in part, by certain arithmetical progressions. A notable feature is that if a representation is possible it may be given by values of the indeterminates all different from zero, and indeed all positive if desired. The numerical coefficients are functions of the solutions of the pure ternary cubic with coefficients 1, 1, $ab(a+b)$, where a, b are any given rational integers, equated to zero. As is well known, the equation mentioned has an infinity of solutions if ab is different from zero. By the processes of the paper we derive from this an infinity of forms of degrees 3 or higher such that all numbers in certain arithmetical progressions are represented in the manner stated. The forms are in 3 or more indeterminates. (Received March 5, 1937.)

243. Mr. Victor Elconin: *Characteristic differential properties of abstract differentiable groups.*

Let Δ be an open set in a Banach space; u is in Δ; ab and a' are functions in Δ. If (1): Δ is a differentiable group with respect to ab, a', u, then linear operators A_a, B_b, with inverses A'_a, B'_b, exist such that (2): $d_\xi{}^a ab = A'_{ab}A_a\xi$, $ub = b$; (3): $d_\eta{}^b ab = B'_{ab}B_b\eta$, $au = u$; (4): $d_\xi{}^a a' = -A'_{a'}A_{a'a}B'_{a'a}B_a\xi$, $u' = u$ (for notations and known results see A. D. Michal and Victor Elconin, American Journal of Mathematics, vol. 59, pp. 129–143). Conversely, if (2), (3), (4) hold and (5): the solution of (2) is unique for each b, or (6): the solution of (3) is unique for each a, then (1) holds. Suppose (1), (5), (6). Then Δ is abelian if and only if $d_\xi{}^a a' = -A'_{a'}A_a\xi$. If moreover $d_\eta{}^a A_a\xi$ or $d_\xi{}^b B_b\eta$ exists, continuous in a or in b respectively, then either a structural function of ab is identically zero or Δ is not abelian. The result still holds if Δ is an integral semi-group with respect to ab, u. (Received March 6, 1937.)

244. Mr. Victor Elconin: *The principal differential.*

Let the domain and range of $f(x)$ be in Banach spaces. Let x be an interior point of the domain of $f(x)$, so that $\Delta_\xi^x f(x) = f(x+\xi) - f(x)$ exists if $\|x\|$ is sufficiently small. The *principal differential*, $p_\xi^x f(x)$ of $f(x)$ is continuous in ξ, satisfies $p_{t\xi}^x f(x) = t^n p_\xi^x f(x)$ for some n and all $t \geqq 0$, and for any $\epsilon > 0$, $\delta > 0$ exists such that $\|\Delta_\xi^x f(x) - p_\xi^x f(x)\| \leqq \epsilon \|\Delta_\xi^x f(x)\|$, so that $(1-\epsilon)\|\Delta_\xi^x f(x)\| \leqq \|p_\xi^x f(x)\| \leqq (1+\epsilon)\|\Delta_\xi^x f(x)\|$, if $\|\xi\| < \delta$. Moreover, for some $\mu \geqq 0$, $\|p_\xi^x f(x)\| \leqq \mu \|\xi\|^n$. Hence $p_\xi^x f(x) = \lim_{t \to 0} 1/t^{-n} \Delta_{t\xi}^x f(x)$, from which $p_\xi^x f(x)$ and n, the *degree* of $p_\xi^x f(x)$, are unique if they exist. If n is the smallest integer such that $(d_\xi^x)^n f(x)$ exists $\not\equiv 0$, then $p_\xi^x f(x)$ exists equal to $(d_\xi^x)^n f(x)/n!$, where $d_\xi^x f(x)$ is the Frechet differential of $f(x)$. The principal differential algorithm is developed and applied to the theory of twisted curves in Banach spaces. (Received March 6, 1937.)

245. Dr. Alvin Sugar: *Additive number theory for the general cubic polynomial.*

The purpose of this paper is to prove that every positive integer is a sum of $m+n+3$ values of $m(x^3-x)/6+n(x^2-x)/2+x$, for m positive and n nonnegative. (Received March 8, 1937.)