Les Théorèmes de la Moyenne pour les Polynomes. By J. Favard. (Actualités Scientifiques et Industrielles; Exposés sur la théorie des fonctions, publiés sous la direction de Paul Montel.) Paris, Hermann, 1936. 51 pp.

The study of the location in the complex plane of the roots of a given polynomial is an interesting and important topic, but perhaps even more interesting geometrically is the study of the mutual relations between the roots of two related polynomials; for instance, one polynomial may be the derivative of the other, or the coefficients of the two polynomials may satisfy a bilinear relation.

The literature on these topics was already extensive at the end of the last century, but has increased enormously since 1900 and is still growing rapidly. At the center of the modern theory lies the theorem of Grace (1902): *If two polynomials*

$$a_0 + C_{n,1}a_1x + C_{n,2}\,a_2x^2 + \cdots + a_nx^n,$$
$$b_0 + C_{n,1}b_1x + C_{n,2}\,b_2x^2 + \cdots + b_nx^n,$$

are apolar:

$$a_0b_n - C_{n,1}a_1b_{n-1} + C_{n,2}a_2b_{n-2} + \cdots + (-1)^n a_nb_0 = 0,$$

then no circle can separate the roots of one polynomial from the roots of the other.

In the essay under review the author sets forth various modern generalizations to polynomials of the mean value theorem of the differential calculus. In the complex domain an illustration is Grace's application of the theorem already stated: *If a polynomial $p(z)$ of degree n takes equal values in the points $+1$ and -1, then the derivative has at least one root on or within the circle whose center is the origin and radius* $\operatorname{ctn}(\pi/n)$. Other results in this domain of ideas, clearly presented by Favard, are due to Heawood, Kakeya, Biernacki, Szegö, Montel, Alexander, and others.

In the real domain there presents itself a problem related both to Grace's theorem and the mean value theorem: *Characterize the sets of real constants $c_0, c_1, c_2, \cdots, (c_0>0)$, such that every polynomial with real coefficients*

$$P(z) \equiv a_0 + a_1z + a_2z^2 + \cdots + a_nz^n, \text{ where } a_0c_0 + a_1c_1 + \cdots + a_nc_n = 0,$$

has at least one real root. Favard has already given an elegant answer: *It is necessary and sufficient that there exist a solution Ψ to the problem of moments*

$$\int_{-\infty}^{\infty} x^n d\Psi = c_n.$$

In this essay further interesting results and applications follow in a most natural manner from formulas of mechanical quadratures. The present knowledge on this topic is due in successive stages to Pompeiu, Montel, Biernacki, Tchakaloff, Favard; related theorems involving more general functions are due to Schoenberg. Direct extensions to the complex plane have as yet not been made; but certain known results (Carathéodory) for the case of the circle have some significance.

Favard's exposition is clear and precise, stimulating in its suggestion of unsolved problems, valuable for a number of new methods and results.

There is a crying need for further capable efforts to continue the unification of the entire subject of the location of roots!

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