$a_{112}=a_{121}=a_{212}=1$. The trilinear form associated with this matrix is $x_1y_1z_2+x_1y_2z_1+x_2y_1z_1$, which is equivalent to L under the transformations $x_1=x_2'$, $x_2=x_1'$. Since the matrices of S and K can be taken to be the same, the factorization rank of S is 1. We have proved the following result.

THEOREM. *The factorization ranks of the forms in the sets* (K, S), (R, P, H), *and* (L, Q) *are* 1, 2, *and* 3, *respectively.*

The equivalence of cubics to P, Q, S can be recognized very simply without the use of factorization rank from the theory of my previous Bulletin paper.

ARMOUR INSTITUTE OF TECHNOLOGY

A NOTE ON THE DEGREE OF POLYNOMIAL APPROXIMATION*

BY J. H. CURTISS

Let C be a rectifiable Jordan curve of the finite z plane. We shall say that a function $f(z)$ belongs to the class Lip (C, j, α) if $f(z)$ is regular in the limited region bounded by C (which we shall call the interior of C), if $f(z)$ is continuous in the corresponding closed region, and if the jth derivative of $f(z)$ is also continuous in this closed region and satisfies a Lipschitz condition with exponent α on C:

$$\left| f^{(j)}(z_1) - f^{(j)}(z_2) \right| \leqq M \left| z_1 - z_2 \right|^{\alpha},$$

z_1, z_2 on C. The number α will be positive and not greater than unity. The number j will be a positive integer or zero; we define $f^{(0)}(z)$ to be identically $f(z)$. The object of this note is to establish the following existence theorem.

THEOREM. *Let the point set S consist of a finite number of closed limited Jordan regions of the z plane bounded by the mutually exterior analytic curves* $C_1, C_2, \cdots, C_\lambda$. *Let the functions* $f_1(z), f_2(z), \cdots, f_\lambda(z)$ *belong respectively to the classes*

$$\text{Lip}\,(C_1, j, \alpha), \qquad \text{Lip}\,(C_2, j, \alpha), \qquad \cdots, \qquad \text{Lip}\,(C_\lambda, j, \alpha),$$

* Presented to the Society, December 31, 1935.

and let $f(z)\equiv f_\nu(z)$, *where* z *is on and interior to* C_ν, *for* $\nu=1, 2, \cdots, \lambda$. *Then there exist polynomials* $p_n(z)$ *of respective degrees* n,* $(n=1, 2, \cdots)$, *such that* $f(z)-p_n(z)=O(n^{-j-\alpha})$ *uniformly for* z *on* S.

The theorem for the case in which S is the closed interior of the circle γ: $|z|=1$ is an immediate consequence of certain results of Bernstein and Jackson on approximation by trigonometric sums. For let the function $f(z)$ belong to the class Lip $(\gamma, 0, \alpha)$. If we write $z=re^{i\theta}$ and $f(e^{i\theta})\equiv u(\theta)+iv(\theta)$, it is easily verified that both $u(\theta)$ and $v(\theta)$ satisfy Lipschitz conditions with exponent α. The work of Bernstein and Jackson† proves that there exist two trigonometric sums of the nth order,

$$U_n(\theta) = \frac{1}{2} d_{n,0}a_0 + \sum_{k=1}^{n} d_{n,k}(a_k \cos k\theta + b_k \sin k\theta),$$

$$V_n(\theta) = \frac{1}{2} d_{n,0}g_0 + \sum_{k=1}^{n} d_{n,k}(g_k \cos k\theta + h_k \sin k\theta),$$

such that uniformly for all values of θ, $u(\theta)-U_n(\theta)=O(n^{-\alpha})$ and $v(\theta)-V_n(\theta)=O(n^{-\alpha})$. The numbers a_k and b_k are the Fourier coefficients of the function $u(\theta)$ and the numbers g_k and h_k are the Fourier coefficients of the function $v(\theta)$. Let $f(z) =\sum_{k=0}^{\infty}A_kz^k$, $|z|\leqq 1$. Then

$$U_n(\theta) + iV_n(\theta) = \sum_{k=0}^{n} d_{n,k}A_k e^{ik\theta},$$

and it follows that

$$f(z) - \sum_{k=0}^{n} d_{n,k}A_k z^k = O(n^{-\alpha})$$

uniformly for $|z|\leqq 1$. The extension to positive values of j is readily effected by methods similar to those employed by Jackson‡ in this connection; the details are left to the reader.

* A polynomial in z of degree n is any expression of the form $a_0+a_1z+a_2z^2+\cdots+a_nz^n$; we do not assume a_n to be different from zero.

† Jackson, *The Theory of Approximation*, Colloquium Publications of this Society, vol. 11, 1930, pp. 1–12; in particular, Theorem IV; Bernstein, Mémoires de l'Académie Royale de Belgique, (2), vol. 4 (1912), pp. 88–89.

‡ Jackson, op. cit., pp. 9–11.

We turn to the general case. Let $w=\psi_\nu(z)$ denote an analytic function which maps the interior of the curve C_ν conformally onto the interior of the circle γ: $|w|=1$, and let $z=\psi_\nu^{-1}(w)$ denote the inverse of this function. A number $R>1$ exists such that the analytic curves C_ν' : $\psi_\nu(z)=R$, $(\nu=1, 2, \cdots, \lambda)$, are mutually exterior and such that the function $\psi_\nu(z)$ is regular and univalent in a limited simply connected region containing the curve C_ν', $(\nu=1, \cdots, \lambda)$. By using the fact that the function $\psi_\nu^{-1}(w)$ is regular for $|w|<R$, we may easily show that each of the functions $f_\nu[\psi_\nu^{-1}(w)]$ belongs to the class Lip (γ, j, α). Hence for each value of ν there exist functions $F_{\nu,n}(z)$ that are polynomials in $w=\psi_\nu(z)$ of respective degrees n, $(n=1, 2, \cdots)$, such that

$$(1) \qquad f_\nu(z) - F_{\nu,n}(z) = O(n^{-j-\alpha})$$

uniformly for z on and interior to C_ν. The functions $F_{\nu,n}(z)$, $(n=1, 2, \cdots)$, are regular in a limited simply connected region containing the curve C_ν'. Moreover, since $F_{\nu,n}(z)=O(1)$ uniformly for $|\psi_\nu(z)|=1$, it follows from a result of Faber* that $F_{\nu,n}(z)=O(R^n)$ uniformly for $|\psi(z)|\leqq R$. We define functions $F_n(z)$ as follows: $F_n(z)\equiv F_{\nu,n}(z)$, z on and interior to C_ν', $(n=1, 2, \cdots; \nu=1, 2, \cdots, \lambda)$. Then

$$(2) \qquad F_n(z) = O(R^n)$$

uniformly for z on and interior to C_ν', $(\nu=1, \cdots, \lambda)$.

Now let $\phi=\phi(z)$ denote an analytic function which maps the complement of S conformally (but not necessarily uniformly) onto the region $|w|>1$ so that the point $z=\infty$ corresponds to the point $w=\infty$. We choose a number $\mu>1$ such that the locus Γ: $|\phi(z)|=\mu$ consists of λ analytic curves lying respectively interior to the curves C_ν' but containing the curves C_ν in their respective interiors.† We also select numbers μ_1, $1<\mu_1<\mu$, and r, $\mu_1/\mu<r<1$. The λ components of the locus Γ_1: $|\phi(z)|=\mu_1$ lie respectively interior to those of the locus Γ, but contain the curves C_ν in their respective interiors.

* Faber, Münchner Berichte, 1922, pp. 157–178. See also Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, Colloquium Publications of this Society, vol. 20, 1935, pp. 77–78.

† For the proof that such a number μ exists, see, for instance, Walsh, op. cit., pp. 65–68, where the locus $|\phi(z)|=K\geqq 1$ is described in some detail.

It has been established by Szegö* that there exists on the boundary of S a set of points $z_k^{(m)}$, $(k=1, 2, \cdots, m+1;$ $m=0, 1, 2, \cdots)$, such that

$$(3) \qquad |\omega_m(z)|^{1/m} \to \Delta |\phi(z)|$$

uniformly for z on any closed point set of the complement of S, where Δ denotes the transfinite diameter of S, and where $\omega_m(z)=(z-z_1^{(m)})(z-z_2^{(m)}) \cdots (z-z_{m+1}^{(m)})$. The polynomial of degree m which coincides with the function $F_n(z)$ in the points $z_k^{(m)}$, $(k=1, 2, \cdots, m+1)$, may be written as follows:

$$P_m(z) = \frac{1}{2\pi i}\int_\Gamma \frac{F_n(t)}{t-z}\left(1-\frac{\omega_m(z)}{\omega_m(t)}\right)dt,$$

as may be verified directly; and for z on Γ_1 we have

$$(4) \qquad |P_m(z)-F_n(z)| \leqq \frac{1}{2\pi}\int_\Gamma \left|\frac{F_n(t)}{t-z}\,\frac{\omega_m(z)}{\omega_m(t)}\right| |dt|.$$

By (3) we have

$$(5) \qquad \left|\frac{\omega_m(z)}{\omega_m(t)}\right| < r^m, \qquad (z \text{ on } \Gamma_1,\ t \text{ on } \Gamma),$$

for all values of m sufficiently large. Let d denote the minimum of $|t-z|$ for t on Γ, z on Γ_1, and let l denote the length of Γ. Let $m=qn$, where q is a positive integer such that $r_1=r^qR<1$. From (2), (4), and (5) we obtain

$$(6) \qquad |P_{qn}(z)-F_n(z)| \leqq \frac{M_1R^n}{d}r^{qn}l = M_2r_1^n, \qquad (z \text{ on } \Gamma_1),$$

for all values of n sufficiently large, where M_1 and M_2 are independent of n and z. By the principle of the maximum, (6) holds for z interior to each of the curves which compose Γ_1. When we combine (6) with (1), we have

$$f(z) - P_{qn}(z) = O(n^{-j-\alpha} + r_1^n) = O(n^{-j-\alpha})$$

* Szegö, Mathematische Zeitschrift, vol. 21 (1924), pp. 205–207; see also Walsh and Russell, Transactions of this Society, vol. 36 (1934), pp. 13–28; Walsh, op. cit., pp. 68–75. The result was proved by Fejér (Göttinger Nachrichten, 1918, pp. 319–331) in the case $\lambda=1$, using methods developed by Hilbert (Göttinger Nachrichten, 1897, pp. 63–70).

uniformly for z on S. We now define the polynomials $p_n(z)$ referred to in the statement of the theorem by the following relations: $p_n(z)\equiv 0$, $(n=1, 2, \cdots, q-1)$, $p_{qn'+h}(z)\equiv P_{qn'}(z)$, $(h=0, 1, \cdots, q-1; n'=1, 2, \cdots)$. Since $O(n'^{-j-\alpha})=O[(qn'+h)^{-j-\alpha})]$, it follows that $f(z)-p_n(z)=O(n^{-j-\alpha})$ uniformly for z on S, and the proof is complete.

If the smoothness of the function $f(z)$ on the boundary of S is described in terms of a modulus of continuity instead of in terms of a Lipschitz condition, the above methods may be used to derive a result analogous to Theorem IV on page 12 of Jackson's *The Theory of Approximation*. The following special case is of importance.*

THEOREM. *If the function $f(z)$ is regular in each of the regions contained in S and is continuous and possesses a continuous jth derivative for z on S, then there exist polynomials $p_n(z)$ of respective degrees n, $(n=1, 2, \cdots)$, such that $f(z)-p_n(z)=O(n^{-j})$ uniformly for z on S.*

The theorem may be used to obtain results on the degree of convergence of certain special sequences of functions. For example, let C be an analytic Jordan curve of the finite plane, let $f(z)$ be a function of the class Lip (C, j, α), and let polynomials $L_n(z)$ of respective degrees n, $(n=0, 1, 2, \cdots)$, be defined by the requirement of coinciding with the function $f(z)$ in the points $z_k^{(n)}=\Phi(e^{2\pi ik/(n+1)})$, $(k=1, 2, \cdots, n+1)$, where the function $z=\Phi(w)$ maps the unlimited region bounded by C onto the region $|w|>1$ so that the point $w=\infty$ corresponds to the point $z=\infty$. A previous result of the author† expresses the degree of convergence of the sequence $\{L_n(z)\}$ in terms of that of an arbitrary sequence of polynomials. This result, when combined with the first theorem of the present paper, yields‡ the equations

* The theorem for the case $j=0$ is included in a theorem of Walsh; see Walsh, op. cit., p. 47, Theorem 15.

† J. H. Curtiss, Transactions of this Society, vol. 38 (1935), pp. 458–473; p. 467.

‡ This application of the theorem is analogous to Jackson's application of the results in pp. 1–12 of his book to the determination of the degree of convergence of trigonometric sums interpolating to a real function $F(x)$ in equally spaced points. These sums exhibit the degree of convergence involved in (8)

(7) $$f(z) - L_n(z) = O(n^{-j-\alpha})$$

uniformly for z on an arbitrary closed point set T interior to C and

(8) $$f(z) - L_n(z) = O(n^{-j-\alpha} \log n)$$

uniformly for z on C.

We may also obtain the degree of convergence of certain sequences of Riemann sums which tend to the Cauchy integral of the function $f(z)$ over the curve C. If we write

$$\sigma_n(z) = \sum_{k=1}^{n+1} \frac{f(z_k^{(n)})}{z - z_k^{(n)}} (z_{k+1}^{(n)} - z_k^{(n)}),$$

then it is possible to show that*

(9) $$\sigma_n(z) - 2\pi i L_n(z) = O(\rho^n)$$

uniformly for z on T, where $\rho < 1$. Comparison of (7) with (9) indicates that†

$$\sigma_n(z) - \int_C \frac{f(t)}{t - z} dt = O(n^{-j-\alpha})$$

uniformly for z on T.

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when the function $F(x)$ has properties corresponding to those of a function of class Lip (C, j, α).

W. E. Sewell (this Bulletin, vol. 41 (1935), pp. 111–117) has recently shown that the partial sums of the expansion of a function $f(z)$ which is of class Lip (C, j, α) in the Faber polynomials belonging to the region interior to C also exhibit the degree of convergence involved in (8) for z on C.

* This equation may be obtained by methods used by Curtiss, loc. cit.; see in particular pp. 460–464 and p. 466.

† For further results on degree of convergence of Riemann sums, see Pólya and Szegö, *Aufgaben und Lehrsätze*, vol. 1, 1925, pp. 35–37, 194–195; and see Walsh and Sewell, this Bulletin, vol. 42 (1936), p. 489, Abstract 284.