ON THE GENERATION OF THE FUNCTIONS Cpq AND Np OF LUKASIEWICZ AND TARSKI BY MEANS OF A SINGLE BINARY OPERATION

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Indicating the n "truth-values" of a Lukasiewicz-Tarski logic† by the n numbers $1, 2, \cdots, n$, we define the functions Cpq and Np as follows:

$$
\begin{aligned}
&Cpq = 1, \text{ when } p \geqq q,\\
&Cpq = q - p + 1, \text{ when } p < q,\\
&Np = n - p + 1.
\end{aligned}
$$

Thus, for example, for $n = 3$ we have

C	1	2	3
1	1	2	3
2	1	1	2
3	1	1	1

p	Np
1	3
2	2
3	1

I shall denote a Lukasiewicz-Tarski logic of n truth-values by L_n.

In this paper I define,‡ in terms of Cpq and Np, a function E_ipq such that, in each L_n, Cpq and Np are in turn definable in terms of $E_{n-2}pq$. The function E_ipq is defined by means of the following series of definitions.

DEFINITION 1. $A_0p = p, A_{i+1}p = CNpA_ip$.
DEFINITION 2. $B_0p = Np, B_{i+1}p = CpB_ip$.
DEFINITION 3. $D_ip = CA_ipNCpNB_ip$.
DEFINITION 4. $E_ipq = CpD_iq$.

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† For a general discussion of this logic, see Lewis and Langford, *Symbolic Logic*, pp. 199–234.

‡ D. L. Webb has recently found (*The generation of any n-valued logic by one binary operation*, Proceedings of the National Academy of Sciences, vol. 21 (1935), pp. 252–254) a binary operation by means of which it is possible to generate any operation of any n-valued logic. His operation, however, cannot be defined in terms of Cpq and Np except when $n = 2$. This can be seen from the fact that the operations Cpq and Np are class-closing on the elements $1, n$; whereas the operation found by Webb has not this property.

In terms of $E_i pq$ I define certain other functions as follows:

DEFINITION 5. $F_i p = E_i E_i pp E_i E_i pp E_i pp$.
DEFINITION 6. $M_i p = E_i p F_i p$.
DEFINITION 7. $I_i pq = E_i p E_i F_i qq$.

I shall now show that, in L_n, $M_{n-2} = Np$, and $I_{n-2}pq = Cpq$; hence that, in L_n, Cpq and Np are definable in terms of the single binary operation $E_{n-2}pq$.

THEOREM 1. *For every n in L_n we have*

$$A_{n-2}n = n, \qquad \text{and} \qquad A_{n-2}p = 1 \text{ for } p \neq n.$$

PROOF. I prove the first part of the theorem by mathematical induction on i. By Definition 1, $A_0 n = n$. Suppose that $A_k n = n$; then $A_{k+1}n = CNnA_k n = CNnn = Cln = n$. Hence for every i we have $A_i n = n$; so, in particular, $A_{n-2}n = n$.

I prove the second part of the theorem by *reductio ad absurdum*. Suppose, if possible, that the second part of the theorem is false, so that there exists a $p_0 < n$ for which $A_{n-2}p_0 > 1$. I first show that, on this supposition, $A_i p_0 > 1$ for every $i \leqq n-2$; for if we had $A_i p_0 = 1$ we should have $A_{i+1}p_0 = CNp_0 A_i p_0 = CNp_0 1 = 1$, so we should have $A_{n-2}p_0 = 1$, contrary to hypothesis. It can be shown that $A_1 p_0 \leqq n-2$; for from $p_0 < n$ follows $p_0 \leqq n-1$, whence $2p_0 \leqq 2n-2$, whence $2p_0 - n \leqq n-2$; and, since $A_1 p_0 \neq 1$, we have $A_1 p_0 = CNp_0 p_0 = p_0 - (Np_0) + 1 = p_0 - (n - p_0 + 1) - 1 = 2p_0 - n$. It can also be shown that for each k, $(n-2 > k > 1)$, we have $A_{k+1}p_0 < A_k p_0$; for from $p_0 < n$ follows $n - p_0 + 1 > 1$, so $Np_0 > 1$; whence $A_k p_0 - Np_0 + 1 < A_k p_0$, and since $A_{k+1}p_0 \neq 1$, we have $A_{k+1}p_0 = A_k p_0 - Np_0 + 1$. Thus we have

$$A_{n-2}p_0 < A_{n-3}p_0 < \cdots < A_2 p_0 < A_1 p_0 \leqq n - 2.$$

Hence

$$A_{n-2}p_0 \leqq A_1 p_0 - (n-3) \leqq (n-2) - (n-3),$$

and $A_{n-2}p \leqq 1$. But this is contrary to hypothesis. Hence the second part of the theorem is true.

The proof of the following theorem is similar.

THEOREM 2. *For every n in L_n we have*

$$B_{n-2}1 = n, \qquad \text{and} \qquad B_{n-2}p = 1 \text{ for } p \neq 1.$$

THEOREM 3. *For every n in L_n we have*

$$D_{n-2}1 = n, \qquad D_{n-2}n = 1, \qquad D_{n-2}p = p \text{ for } p \neq 1, n.$$

PROOF. By Theorems 1 and 2, and the definitions of Cpq and Np, we have $D_{n-2}1 = CA_{n-2}1NC1NB_{n-2}1 = C1NC1Nn = C1NC11 = C1n = n$, $D_{n-2}n = CA_{n-2}nNCnB_{n-2}n = CnNCn1 = CnNn = Cn1 = 1$. Suppose now that $p \neq 1, n$. Then $D_{n-2}p = CA_{n-2}pNCpNB_{n-2}p = C1NCpN1 = C1NCpn = C1N(n-p+1) = C1[n-(n-p+1)+1] = C1p = p$.

THEOREM 4. *For every $p \neq 1$ in L_n, $E_{n-2}pp = 1$; and $E_{n-2}11 = n$.*

PROOF. If $p \neq 1$, n then, by Theorem 3, $E_{n-2}pp = CpD_{n-2}p = Cpp = 1$. If $p = n$, then $E_{n-2}pp = CnD_{n-2}n = Cn1 = 1$. If $p = 1$, finally, $E_{n-2}pp = C1D_{n-2}1 = C1n = n$.

THEOREM 5. *For every p in L_n, $F_{n-2}p = 1$.*

PROOF. If $p \neq 1$, then, by Theorem 4, we have

$$\begin{aligned} F_{n-2}p &= E_{n-2}E_{n-2}ppE_{n-2}E_{n-2}ppE_{n-2}pp = E_{n-2}1E_{n-2}11 \\ &= E_{n-2}1n = C1D_{n-2}n = C11 = 1. \end{aligned}$$

If $p = 1$, then, again by Theorem 4,

$$\begin{aligned} F_{n-2}p &= E_{n-2}E_{n-2}11E_{n-2}E_{n-2}11E_{n-2}11 = E_{n-2}nE_{n-2}nn \\ &= E_{n-2}n1CnD_{n-2}1 = Cnn = 1. \end{aligned}$$

THEOREM 6. *For every p in L_n, $M_{n-2}p = Np$.*

PROOF. $M_{n-2}p = E_{n-2}pF_{n-2}p = E_{n-2}p1 = CpD_{n-2}1 = Cpn = Np$.

THEOREM 7. *For every p and q in L_n, $I_{n-2}pq = Cpq$.*

PROOF.

$$\begin{aligned} I_{n-2}pq &= E_{n-2}pE_{n-2}F_{n-2}qq = E_{n-2}pE_{n-2}1q \\ &= E_{n-2}pC1D_{n-2}q = E_{n-2}pD_{n-2}q = CpD_{n-2}D_{n-2}q. \end{aligned}$$

But, by Theorem 3, we have $D_{n-2}D_{n-2}q = q$. Hence $I_{n-2}pq = Cpq$.

Thus we have shown that in each L_n it is possible to define in terms of Cpq and Np a function, namely, $E_{n-2}pq$, in terms of which Cpq and Np are again definable.

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