ALEXANDROFF AND HOPF ON TOPOLOGY

Topologie I. By Paul Alexandroff and Heinz Hopf. Berlin, Springer, 1935. xiii+636 pp.

Topology is not a young subject; this year may be described as the two hundredth anniversary of its birth if we agree that it had its beginning in the problem of the seven bridges of Königsberg settled by Euler in 1736. But the systematic development of topology is new; it has only come since the work of Poincaré at the turn of the century. The International Topological Conference held at Moscow last September showed that the subject has attained a definite measure of maturity and a wide range of influence on other branches of mathematics, but that it is still undergoing rapid growth and flux. Just when topological activity seems to be slackening some new point of view sets it seething again; in the past year we have had an example of this in the "dual cycles" of Alexander and Kolmogoroff.

Books on topology are so few that the appearance of a new one is an important event. The volume now before us is especially impressive, not only for its own size and thoroughness, but because it is the first of three volumes which together are intended to give a detailed survey of topology as a whole. The authors are distinguished geometers who have been in close contact with the various topological centers in Europe and America. They began their difficult task several years ago at the suggestion of Courant, under whose editorship the book appears as the forty-fifth of the familiar Springer series. Many of us have known about the project and have expectantly awaited the finished work.

In their attempt to treat topology as a whole the authors have steered a middle course between point-set theory on the one hand and algebraic topology on the other. They do this quite easily since in their own investigations they both have used combinatorial methods to solve set-theoretic problems; no better example of such blending could be cited than Alexandroff's theory of dimension. As the central concept for the first volume they have chosen the (euclidean) *polyhedron*, a subset of multi-dimensional euclidean space which can be partitioned into flat convex cells in an orthodox manner. The polyhedron holds in their program an intermediate position between the generality of an *abstract* space and the special character of a *manifold;* by appropriate approximation, for example, one passes from polyhedra to compact metric spaces, and by suitable specialization one passes from a polyhedron to a manifold. In accordance with this plan the second volume is to deal with set-theoretic questions, such as dimensionality, and the third with manifolds.

The book (that is, the first volume) falls into four parts. Part I is a hundred-page introduction to abstract-space topology; much of this has no bearing on the subsequent study of polyhedra but it does have an essential place in the general program of the three volumes. The first chapter begins with various methods of topologization, closure, limits, distance, neighborhoods, and so on, then passes to separation and countability axioms, and finally to the Urysohn theorem that a normal space with countable basis can be immersed in Hilbert space. The second chapter treats the fundamental properties of compact and bicompact spaces. The authors exploit to interesting advantage the notion of a decomposition of a space into mutually exclusive closed sets.

In Part II we encounter the fundamental machinery of combinatorial topology set up on abstract simplicial complexes, finite or countably infinite. Chapter 3 introduces polyhedra and their subdivisions into convex cells as a concrete basis from which to pass to abstract complexes. Chapters 4 and 5 develop the homology theory of absolute (abstract) complexes in strict group terms, the theory of Abelian groups prerequisite for this being given in an appendix. For chains and cycles the authors use coefficients drawn from a general Abelian (additive) group, assuming only when necessary that the coefficient group is a ring. Moreover they introduce two coefficient groups at once, one group for cycles and a second one, which contains the first, for homologies; this is a generalization of cycles with integer coefficients which do not bound even with division allowed. Detailed consideration is given to the special properties of homology over the rings of integers, of rationals, and of residues mod m, and over the Pontrjagin group of rationals mod 1, which is proving such a useful idea nowadays. The invariance of homology under subdivision is established. Chapter 6 discusses the consolidation of simplexes into more general combinatorial units and relates it to the subdivision of polyhedral complexes. A final chapter (7) deals with closed complexes (circuits) and with the homology theory of sums and (cartesian) products of complexes.

In Part III we return to (euclidean) polyhedra to begin a comprehensive investigation of the topological invariants which attach to them out of the homology theory now established. Chapter 8 uses simplicial approximation of continuous mapping to collect homotopic mapping-classes into homology classes and so to establish in large measure the topological invariance of homology; the same chapter also introduces continuous (singular) chains and cycles, and the notion of retract. The main proofs of the invariance of homology groups and of dimension are formulated in Chapter 9 in terms of coverings and nerves; much of this applies to compact metric spaces as well as to polyhedra (including infinite polyhedra). Chapter 10 has to do with the decomposition of a euclidean space by a compact metric subspace, the Jordan-Brouwer theorem, and the invariance of open regions.

Part IV continues the application of homology theory to the study of polyhedra—first to the relative position a polyhedron may have in euclidean space, and then to properties of mappings of polyhedra into themselves or into euclidean spaces or spheres. The question of relative position is handled in Chapter 13 by the theory of linking and the Alexander duality theorem. Chapter 14 deals with the Brouwer concept of the degree of a mapping, and with outgrowths thereof. In Chapter 15 we have a discussion of the extension of mappings in euclidean space and of the types of maps of an n-dimensional polyhedron on the n-sphere or on the circle. A very full account of the Lefschetz fixed-point formula as set up for polyhedra by Hopf occupies the last chapter (16). There are two appendices, one on Abelian groups and the other on convex cells in euclidean space.

The authors have carried through the above program with great completeness of detail; they have explored many bypaths, and in some instances they have given alternative demonstrations for basic propositions and alternative approaches to important concepts. Unfortunately this extreme comprehensiveness coupled with a tendency to be somewhat discursive renders the book formidable for beginners and not exactly suited to straightforward reading. But a reader with some topological experience should be able to make his way about in the book and benefit from its encyclopaedic nature without being overwhelmed. He will find his task facilitated by a detailed table of contents at the beginning of each chapter, a good general index, and abundant cross-references (but not to page numbers, unfortunately). There are also many helpful figures and examples.

The book contains no real bibliography; it has merely a list of topological texts and one of works which bear directly on individual sections of the book and influenced their composition. References to the literature through footnotes have been reduced to a minimum. Certain concepts and proofs have been designated by the names of their originators, but otherwise the authors have resolutely refrained from attempting to trace notions back to their sources. The introduction to the book embraces a short history of the development of topology and a survey of the relations of topology to neighboring branches of mathematics, as well as a discussion of the authors' program. Very appropriately the book is dedicated to L. E. J. Brouwer.

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