CERTAIN NON-INVOLUTORIAL CREMONA TRANSFORMATIONS OF HYPERSPACE

BY HARRIET F. MONTAGUE

1. *Introduction.* The problem here presented was suggested by a paper by Maria Miglio.* A four page synthetic outline given there for S_4 has been enlarged upon and extended to S_r in the present paper.

2. *Definition of the Transformation.* A non-involutorial Cremona transformation in S_r is set up as follows. A (1, 1) correspondence is established between the elements of a pencil of primals on a base o_1 and the points of a rational curve f_n of order n. A (1, 1) correspondence is also established between the elements of a pencil of primals on a base o_2 and the points of f_n. Any point P in S_r determines with o_1 an element of the first pencil and associated with this element there is a point A on f_n. Associated with A there is an element of the second pencil on o_2. The intersection of this element with the line AP gives P', the image of P in the transformation. The inverse transformation proceeds in a similar way starting with the pencil of elements on o_2.

This transformation is examined as applied to the following cases:

Case A. Pencil of primes on o_1, pencil of primes on o_2.

Case B. (a) Pencil of quadric primals on o_1, pencil of primes on o_2.

(b) Pencil of quadric primals on o_1, pencil of quadric primals on o_2.

Case C. (a) Pencil of cubic primals on o_1, pencil of primes on o_2.

(b) Pencil of cubic primals on o_1, pencil of quadric primals on o_2.

(c) Pencil of cubic primals on o_1, pencil of cubic primals on o_2.

Case D. Pencils of primals of higher order with f_n a line f_1.

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* Maria Miglio, *Nell' S_4 una classe di trasformazioni birazionali,* Catania Accademia Gioenia Atti, (5), vol. 18 (1932) [Mem. 19], pp. 1–22.

3. *Case* A. The transformation is a T_{n+2} and the inverse is of the same order. The locus of invariant points is the locus of intersection of corresponding elements of the two pencils, as is found in all the cases following. The fundamental elements of the transformation are:

(1) The $n+1$ points of f_n through which pass the associated primes on o_1. Their images are the corresponding primes on o_2.

(2) The $n+1$ intersections of primes on o_2 which pass through their associated points on f_n with the corresponding primes on o_1. Their images are the $n+1$ primes on o_2 which pass through their associated points.

(3) The base o_1. Its image is the projection of a ruled surface of order $n+1$, containing o_1 once, o_2 n times, from the common part of o_1 and o_2 as vertex.

The residual intersection of two general homaloids, apart from the fundamental elements, is of order $n+2$. In S_3, C_{n+2} has $n+1$ points on o_1, one point on each of the simple fundamental lines, and $n+1$ points on f_n.

4. *Case* B. In using a pencil of quadric primals, the point associated with a particular quadric must lie on that quadric. If we have f_n a part of the base of the pencil of quadrics, in S_3, $n=1$, 2, or 3; in S_r, $r>3$, f_n can be of any order.

(a) A pencil of quadric primals and a pencil of primes gives T_{2n+3} with the inverse T_{n+3}^{-1}.

The image of the base of the pencil of quadrics, a manifold M_4, is a primal. If M_4 does not contain f_n, M_4 meets f_n in $2n-1$ points. In S_3, the image of C_4 is a surface of order $2n+5$, containing o_2 to multiplicity $2n+1$. If f_n is a line f_1, f_1 counts once in the image of C_4. On the other hand, if f_1 is a part of the base, the image of the residual C_3 contains o_2 once and f_1 twice.

Whether f_n is a part of the base M_4 or not, the point A lies on its associated quadric. In S_3 the two generators of the associated quadric which pass through A generate a ruled surface of order $2n+3$ as A moves along f_n. This R_{2n+3} is a principal surface in the transformation.

(b) Two pencils of quadric primals give a T_{2n+4} with an inverse of the same order. The properties of this transformation may be found in a manner similar to that of (a).

5. *Case* C. When a pencil of cubic primals is employed, it is

necessary for a point of f_n to be a double point on its associated primal. Then every other point of f_n is a simple point of the primal. The transformation is possible only when f_n is of order $n=1, 2, 3$. The discussion is given for S_3, but the general work may be extended to S_r, $r>3$. The author has made the analytic extension, but omits it here for the sake of simplicity. The properties of transformations (b) and (c) may be found in a manner similar to that of (a).

(a) A pencil of cubic primals and a pencil of primes in S_3. T_{3n+4}; T_{n+4}^{-1}.

(1) $n=1$. T_7; T_5^{-1}. In S_3, the line f_1 is contained twice in the base of the pencil of cubic surfaces, since the surfaces of the pencil touch at every point of f_1, leaving a residual C_7. C_7 meets f_1 in 4 points. The image of each point of C_7 is a conic. The complete image of C_7 is a surface j_{10}' which contains f_1 counted four times and o_2 counted three times, that is, $C_7 \sim j_{10}'$; $o_2^3 f_1^4$.

The image of f_1 is f_1 itself. There are two points of f_1 through which pass their associated planes on o_2. The images of these two points in the inverse transformation are the cubic surfaces associated with these points respectively.

There are two plane cubic curves formed by the intersection of the two planes through their associated points with the corresponding cubic surfaces. These curves are part of the base of the pencil of cubic surfaces. The six lines on any cubic surface which can be drawn from the associated point A generate a ruled surface of order 8, as A moves along f_1, and this R_8 is a principal surface. Every plane through f_1 intersects R_8 in a residual C_3. The curve C_3 breaks up into the three lines joining in pairs the three points of C_7 not on f_1 in the plane.

Two general homaloids of the transformation meet in a variable C_5. The residual intersection is $C_7^4 f_1^{10} \gamma_3 \delta_3$ where γ_3 and δ_3 are the two plane cubic curves mentioned above.

(2) $n=2$. T_{10}; T_6^{-1}. In S_3, the conic f_2 counts twice in the base of the pencil of cubic surfaces, leaving a residual C_5. The plane of f_2 meets C_5 in 5 points, all on f_2. $C_5 \sim j_{10}'$; o_2^5.

The image of any point P on f_2 is a conic passing through P and K, the intersection of o_2 and the plane of f_2, and through the three points of f_2 through which pass the associated planes on o_2. The totality of these conics is the image of f_2. The images, in the inverse transformation, of the three fundamental points of

f_2 are the corresponding cubic surfaces. The ruled surface, analogous to R_8 of the previous case, is, in this case, R_{11}.

(3) $n=3$. T_{13}; T_7^{-1}. In S_3, f_3 counts twice in the base of the pencil of cubic surfaces, leaving a residual C_3. The curve C_3 breaks up into three lines which are bisecants of f_3. $C_3 \sim j_6'$; o_2^3. The image of f_3 is of order 4, containing o_2 twice. There are 4 points on f_3 through which pass associated planes on o_2. Their images are the corresponding cubic surfaces. The principal ruled surface in this case is R_{14}.

(b) A pencil of cubic primals and a pencil of quadric primals give T_{3n+5}, $n=1, 2, 3$, with the inverse T_{2n+5}^{-1}.

(c) Two pencils of cubic primals give T_{3n+6}, $n=1, 2, 3$, with an inverse of the same order.

6. *Case* D. If f_n is taken as a line f_1, it can be of any multiplicity on a surface of any order. Consider a pencil of primals of order s on o_1, and a pencil of primals of order t on o_2. Take f_1 such that a primal on o_1 has an $(s-1)$-fold point at its associated point of f_1 and a primal on o_2 has a $(t-1)$-fold point at its associated point. Then a general point of f_1 is $(s-2)$-fold on every primal of the first pencil, and $(t-2)$-fold on every primal of the second pencil. We obtain a T_{2s+t} with the inverse T_{2t+s}^{-1}.

The locus of invariant points is, as before, the locus of intersection of corresponding elements of the two pencils. In S_3, the bases of the two pencils do not intersect. In S_r, $r>3$, they do, and any point on the common part is invariant under the transformation.

In S_3, the residual part of the base is C_{3s-2} meeting f_1 in $3s-5$ points. The image of C_{3s-2} is of order 6 plus the multiplicity of f_1. If the elements of the second pencil are planes, $C_{3s-2} \sim j_{3s+1}'$; $o_2^3 f_1^{3s-5}$. If the elements of the second pencil are surfaces of order t, the cone joining C_{3s-2} to a point A on f_1 is met by the associated surface of the second pencil in a conical curve of order $(3s-2)t$, the image of C_{3s-2} for the particular surface associated with A.

In S_3, if f_1 is an $(s-1)$-fold line on the first base, the residual part of the base is C_{2s-1}. There is one point of C_{2s-1} not on f_1 in every plane through f_1. The image of C_{2s-1} is of order 2 plus the multiplicity of f_1. If the elements of the second pencil are planes, $C_{2s-1} \sim j_{2s}'$; o_2. If the elements of the second pencil are surfaces of order t, the image of C_{2s-1} for a particular surface is a conical curve of order $(2s-1)t$.

In S_r, the image of any point on the base M_{s2} or M_{t2} is a conic. The lines joining points of M to a particular point of f_1 form a conical primal with a point vertex. It is met by the corresponding primal of the second pencil in a manifold $M^{r-2}_{(3s-2)t}$ or $M^{r-2}_{(2s-1)t}$ according as there is contact or not.

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NOTE ON SOME EQUATIONS WITHOUT AFFECT*

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A numerical equation of degree greater than 4 certainly cannot be solved by radicals if it is "without affect"; that is, if its Galois group is the symmetric group. Hence it is of interest to construct explicitly such equations. A number of such constructions have been developed,† many of them intrinsically related to certain prime-ideal decompositions. Hence the Newton polygon construction for prime ideals and the related Eisenstein irreducibility criterion are relevant, and can be used systematically to give new proofs for several known constructions (Theorem 2) and for some new equations without affect (Theorems 1 and 2 and generalizations). The advantages lie in the uniform procedure and in the ease of the explicit construction of Theorem 1.

THEOREM 1. *Let p, q, and r be rational primes and construct*

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n, \qquad (n \geqq 4), \tag{1}$$

with rational integral coefficients a_i such that: (I) *each a_i is divisible by r, but a_n is not divisible by r^2;* (II) *each a_i is divisible by q, and a_n but not a_{n-1} is divisible by q^2;* (III) *the highest power e_i such that a_i is divisible by p^{e_i} satisfies*

$$e_1 \geqq 1, \qquad e_2 = 1, \qquad e_3 \geqq 2, \qquad e_i - e_{i-1} > e_{i-1} - e_{i-2}, \tag{2}$$

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* Presented to the Society, April 10, 1936, and subsequently extended.

† Ph. Furtwängler, *Ueber Kriterien für irreduzible und für primitive Gleichungen und über die Aufstellung affektfreier Gleichungen*, Mathematische Annalen, vol. 85 (1922), pp. 34–40.