[(5), 9.]	$r \mathrel{-\!\!3} s. = .i$	(6)
[(4), (6)]	$p \mathrel{-\!\!3} q. = .r \mathrel{-\!\!3} s$	(7)
[11.03]	$(7) = (1)(2)$	(8)
[(7), (8)]	$(1)(2)$	(9)
[11.2]	$(1)(2) \mathrel{-\!\!3} (1)$	(10)
[12.17]	$(1)(2) \mathrel{-\!\!3} (2)$	(11)
[(9), (10)]	(1)	
[(9), (11)]	(2) .	

The paradox stated above is a particular case of Theorem 10, and therefore requires no further proof.

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THE BETTI NUMBERS OF CYCLIC PRODUCTS

BY R. J. WALKER

1. *Introduction.* In a recent paper† M. Richardson has discussed the symmetric product of a simplicial complex and has obtained explicit formulas for the Betti numbers of the two- and three-fold products. Acting on a suggestion of Lefschetz, we define a more general type of topological product and apply Richardson's methods to compute the Betti numbers of a certain one of these, the "cyclic" product.

2. *Basis for m-Cycles of General Products.* Let S be a topological space and G a group of permutations on the numbers $1, \cdots, n$. The *product of S with respect to G*, $G(S)$, is the set of all n-tuples $(P_1, \cdots, P_n)$ of points of S, where $(P_{i_1}, \cdots, P_{i_n})$ is to be regarded as identical with $(P_1, \cdots, P_n)$ if and only if the permutation $\binom{1 \cdots n}{i_1 \cdots i_n}$ is an element of G. A neighborhood of $(P_1, \cdots, P_n)$ is the set of all points $(Q_1, \cdots, Q_n)$ for which Q_i belongs to a fixed neighborhood of P_i. It is not difficult to verify that the

† M. Richardson, *On the homology characters of symmetric products*, Duke Mathematical Journal, vol. 1 (1935), pp. 50–69. We shall refer to this paper as R.

Hausdorff axioms hold for this definition of neighborhood, and hence that $G(S)$ is a topological space. In particular, if G is the identity or the symmetric group, $G(S)$ is, respectively, the direct or the symmetric product of S. If G is the cyclic group on n elements we shall call $G(S)$ the n-fold *cyclic* product of S.

The space $G(S)$ can be obtained in another manner. Let S^n denote the n-fold direct product of S. Then each element $\binom{1 \cdots n}{i_1 \cdots i_n}$ of G gives rise to an automorphism of S^n which carries $(P_1, \cdots, P_n)$ into $(P_{i_1}, \cdots, P_{i_n})$. By identifying points which are images of each other under the group of automorphisms we evidently obtain a space homeomorphic to $G(S)$.

Now let K be a simplicial complex, K^n its direct product, and $k=G(K)$ its product with respect to the group G of degree n and order r. We then have r automorphisms T_λ of K^n, and a continuous, single-valued transformation Λ of K^n into k, such that†

$$\Lambda T_\lambda = \Lambda. \tag{1}$$

Richardson has shown‡ that K^n and k can be subdivided into simplexes in such a fashion that the transformations T_λ and Λ are simplicial. We can therefore operate with them on chains of K^n. If E and e are simplexes of K^n and k, respectively, such that $e=\Lambda E$, we define the operator Λ' by $\Lambda' e=\sum_\lambda T_\lambda E$. We have then

$$\Lambda\Lambda' e = re, \tag{2}$$

$$\Lambda'\Lambda E = \sum_\lambda T_\lambda E. \tag{3}$$

We also find that T_λ, Λ, and Λ' preserve boundaries and hence homologies.

The principal theorem of Richardson,§ concerning the Betti numbers of k, is stated in terms of matrices. For actual computation we find it easier to work with the cycles themselves, and so we shall state and prove the theorem in a slightly different form.

† In the expression for the product of two transformations, the transformation represented by the right-hand symbol is to be applied first.

‡ R, pp. 51 and 53.

§ R, p. 52.

THEOREM 1. *Let* $\{\Gamma^i\}$ *be an independent basis, with respect to homology, for m-cycles, with rational coefficients, of* K^n, *such that* $T_\lambda \Gamma^i = \pm \Gamma^{i_\lambda}$, $(\lambda = 1, \cdots, r)$; *and let* $\{\overline{\Gamma}^\alpha\}$ *be a maximal subset of* $\{\Gamma^i\}$ *such that*

$$\text{(a)} \qquad T_\lambda \overline{\Gamma}^\alpha \neq \pm \overline{\Gamma}^\beta, \qquad (\alpha \neq \beta),$$

$$\text{(b)} \qquad T_\lambda \overline{\Gamma}^\alpha \neq - \overline{\Gamma}^\alpha,$$

for any λ. *Then* $\{\gamma^\alpha\} = \{\Lambda \overline{\Gamma}^\alpha\}$ *is an independent basis with respect to homology for the m-cycles of k.*

PROOF. (i) The γ^α are independent. For suppose that we have $\sum_\alpha x_\alpha \gamma^\alpha \sim 0$, that is, $\sum_\alpha x_\alpha \Lambda \overline{\Gamma}^\alpha \sim 0$. Then

$$\Lambda' \sum_\alpha x_\alpha \Lambda \overline{\Gamma}^\alpha = \sum_\alpha x_\alpha \Lambda' \Lambda \overline{\Gamma}^\alpha = \sum_{\alpha,\lambda} x_\alpha T_\lambda \overline{\Gamma}^\alpha \sim 0,$$

by (3). Now if $T_\lambda \overline{\Gamma}^\alpha = \epsilon \Gamma^i$, $\epsilon = \pm 1$, we cannot have $T_\mu \overline{\Gamma}^\alpha = -\epsilon \Gamma^i$, for this would imply

$$T_\mu^{-1} T_\lambda \overline{\Gamma}^\alpha = \epsilon T_\mu^{-1} \Gamma^i = -\epsilon^2 \overline{\Gamma}^\alpha = -\overline{\Gamma}^\alpha,$$

contrary to condition (b). Similarly, from (a), we cannot have $T_\mu \overline{\Gamma}^\beta = \pm \Gamma^i$, $\beta \neq \alpha$. Hence with each such Γ^i there is associated an ϵ_i, a $\overline{\Gamma}^\alpha$, and s_i values of λ for which $T_\lambda \overline{\Gamma}^\alpha = \epsilon_i \Gamma^i$. If the last homology is now written in terms of the basis $\{\Gamma^i\}$, the coefficient of Γ^i will be $\epsilon_i s_i x_\alpha$. Since the Γ^i are independent, $\epsilon_i s_i x_\alpha = 0$, and therefore every $x_\alpha = 0$.

Use was made of the properties of the rational coefficients only in the last step of each part of the proof. Now the s_i introduced in (i) are factors of r, for the T_λ for which $T_\lambda \overline{\Gamma}^\alpha = \epsilon_i \Gamma^i$ evidently form a coset of the subgroup which leaves $\overline{\Gamma}^\alpha$ invariant. It follows that the theorem will hold for any coefficient group in which each element has a unique rth part; in particular for the group of residues modulo a number prime to r.

(ii) $\{\gamma^\alpha\}$ is a basis. We note first that since the set $\{\overline{\Gamma}^\alpha\}$ is maximal every Γ^i is of one of the two forms $T_\lambda \overline{\Gamma}^\alpha$ or $\tilde{\Gamma}^j$, where for each j there is a λ_j such that $T_{\lambda_j} \tilde{\Gamma}^j = -\tilde{\Gamma}^j$. Also, $\Lambda \tilde{\Gamma}^j = \Lambda T_{\lambda_j} \tilde{\Gamma}^j = -\Lambda \tilde{\Gamma}^j$, so that $\Lambda \tilde{\Gamma}^j = 0$. Now if γ is any m-cycle of k, $\Lambda' \gamma$ is an m-cycle of K^n, and so

$$\Lambda' \gamma \sim \sum_i x_i \Gamma^i = \sum_{\alpha,\lambda} x_{\alpha\lambda} T_\lambda \overline{\Gamma}^\alpha + \sum_j x_j \tilde{\Gamma}^j.$$

Hence

$$\Lambda\Lambda'\gamma = r\gamma \sim \sum_{\alpha,\lambda} x_{\alpha\lambda}\Lambda T_\lambda \bar{\Gamma}^\alpha + \sum_j x_j \Lambda \tilde{\Gamma}^j = \sum_{\alpha,\lambda} x_{\alpha\lambda}\gamma^\alpha,$$

by (2) and (1). That is,

$$\gamma \sim \sum_{\alpha,\lambda} \frac{x_{\alpha\lambda}}{r}\gamma^\alpha,$$

3. *Betti Numbers of Cyclic Products.* Keeping the notation as before, we let G be the cyclic group on n elements. To compute the mth Betti number of the cyclic product k we must count the number of m-cycles $\bar{\Gamma}^\alpha$. A basis of the type $\{\Gamma^i\}$ used in the theorem is obtained by taking all cycles of the form

$$C_{m_1} \times \cdots \times C_{m_n}, \qquad m_1 + \cdots + m_n = m,$$

C_{m_i} being a member of a basis of m_i-cycles of K.† Following Richardson's procedure, we obtain

$$\begin{aligned} T_\lambda(C_{m_1} \times \cdots \times C_{m_\lambda} \times C_{m_{\lambda+1}} \times \cdots \times C_{m_n}) \\ = (\dot{-}\,1)^{\epsilon_\lambda} C_{m_{\lambda+1}} \times \cdots \times C_{m_n} \times C_{m_1} \times \cdots \times C_{m_\lambda}, \end{aligned}$$

where

$$\begin{aligned} \epsilon_1 = m_1m_2 + \cdots + m_1m_n = m_1(m - m_1) &= mm_1 - m_1^2 \\ &\equiv mm_1 - m_1 \qquad \pmod 2 \\ &= (m-1)m_1, \end{aligned}$$

and by induction

$$\epsilon_\lambda \equiv (m-1)(m_1 + \cdots + m_\lambda) \qquad \pmod 2.$$

Let q be a factor of n, $n = qs$, and consider all Γ^i which are invariant, to within change of sign, under G_q, the cyclic subgroup of G of order q. They necessarily have the form

$$\begin{aligned} \Gamma_q = (C_{m_1} \times \cdots \times C_{m_s}) \times (C_{m_1} \times \cdots \times C_{m_s}) \times \cdots \\ \times (C_{m_1} \times \cdots \times C_{m_s}), \end{aligned}$$

there being q identical sets of factors. We must have $q(m_1+$

† S. Lefschetz, *Topology*, p. 228.

$\cdots + m_s) = m$; that is, to have a Γ_q, q must be a factor of m and hence of (m, n), the highest common factor of m and n. If t is a proper multiple of q and a factor of (m, n), it is easily seen that a Γ_t is also a Γ_q. We denote by Γ_q^* any Γ_q which is not such a Γ_t, and by $A_{m,q}$ the number of Γ_q^*. The total number of Γ_q is then $\sum_t A_{m,t}$, the summation being over all values of t which are multiples of q and factors of (m, n). But the number of Γ_q is evidently equal to the number of possible combinations of the form $C_{m_1} \times \cdots \times C_{m_s}$, $m_1 + \cdots + m_s = m/q$, and this is exactly $R_{m/q}(K^s)$. Hence

$$\sum_t A_{m,t} = R_{m/q}(K^{n/q}),$$

and from these equations we can obtain the $A_{m,q}$ step by step starting with $q = (m, n)$, or directly by the use of the Dedekind inversion formula.

Now

$$T_s \Gamma_q = (-1)^{(m-1)(m_1 + \cdots + m_s)} \Gamma_q = (-1)^{(m-1)m/q} \Gamma_q,$$

and so if m is even and m/q is odd, Γ_q is a cycle of the type $\tilde{\Gamma}^j$ of Theorem 1 and is not counted among the $\overline{\Gamma}^\alpha$. We therefore put

$$B_{m,q} = \begin{cases} 0, & \text{if } m \text{ is even and } m/q \text{ is odd}, \\ A_{m,q} & \text{otherwise}. \end{cases}$$

Consider the s cycles $\Gamma_q^*, T_1\Gamma_q^*, \cdots, T_{s-1}\Gamma_q^*$. If any two of these are equal, say $T_i\Gamma_q^* = T_j\Gamma_q^*$, $(i > j)$, then Γ_q^* is invariant, to within change of sign, under the subgroup generated by $T_j^{-1}T_i = T_{i-j}$, and hence under the minimal subgroup containing G_q and T_{i-j}. Since $i - j < s$, T_{i-j} is not an element of G_q and therefore this subgroup is a G_t with t a proper multiple of q, contrary to the definition of Γ_q^*. It follows that there are exactly $s = n/q$ distinct transforms of each of the $B_{m,q}$ cycles Γ_q^*, and so we can pick out $(q/n)B_{m,q}$ of the Γ_q^* which are not transformable into one another and which can therefore be included among the $\overline{\Gamma}^\alpha$ of Theorem 1. Since the cycles Γ_q^* for different values of q are not transformable into one another and since every Γ^i is a Γ_q^* for some q, we have the following result.

THEOREM 2.

$$R_m(k) = (1/n)\sum_q qB_{m,q},$$

the summation being over all factors of (m, n).

The following special cases may be of interest.

COROLLARY 1. *If* n *is an odd prime*

$$R_m(k) = \begin{cases} (1/n)R_m(K^n), & \text{if } (m, n) = 1, \\ (1/n)[R_m(K^n) - R_s(K)] + R_s(K), & \text{if } m = ns. \end{cases}$$

COROLLARY 2. *If* p *is an odd prime and* $n = p^\alpha$, $m = p^\beta m_1$, $(m_1, p) = 1$, *and* $\gamma = \min \alpha, \beta$,

$$R_m(k) = \frac{p-1}{n}\left[\frac{1}{p-1} R_m(K^n) + \sum_{i=1}^{\gamma} p^{i-1} R_{m/p^i}(K^{n/p^i})\right].$$

COROLLARY 3. *If* $R_0(K) = 1$, *then* $R_1(k) = R_1(K)$.

4. *Remark.* The methods used on the cyclic product can evidently be used to compute the Betti numbers of a product with respect to an arbitrary group. In general, however, the resulting formulas are too complicated to be of interest.

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